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# METRIC-AFFINE THEORIES OF GRAVITY: FOUNDATIONS AND APPLICATIONS


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# Abstract

The purpose of this thesis is to outline equivalent representations of Gravity in the framework of Metric-Affine geometries. To be specific, we focus on the so-called Trinity of Gravity: General Relativity (GR), constructed upon the metric tensor and based on curvature $R$; the Teleparallel Equivalent of General Relativity (TEGR), formulated in terms of torsion $T$ and relying on tetrads and spin connection; and the Symmetric Teleparallel Equivalent of General Relativity (STEGR), built on non-metricity $Q$ and constructed from the metric tensor and an affine connection. Teleparallel formulations provide a gauge-based description of gravity and recover the foundational principles of GR, including the Equivalence Principle and general covariance. Although dynamically equivalent, namely their Lagrangians differ only by a boundary term and lead to the same field equations and solutions, the three theories offer distinct conceptual interpretations. We investigate the role of the Equivalence Principle, showing that while it is fundamental in GR, it is recovered rather than postulated in TEGR and STEGR, potentially leading to differences in empirical content. We analyze the motion of test particles, highlighting that the equivalence within the Trinity depends on the matter sector: point-like particles without hypermomentum follow identical trajectories, whereas particles with internal structure may exhibit different free-fall behaviour.

Finally, we discuss the ambiguities arising from the coupling between matter and spacetime geometry in Metric-Affine theories. Both bosonic and fermionic fields present issues; however, we present several approaches to restore a consistent matter coupling in both TEGR and STEGR.

# Summary

# Introduction

Gravity is one of the four fundamental interactions of Nature.
In 1915, Einstein, inspired by studies in *affine geometry* and *metric geometry*, formulated the theory of General Relativity (GR), which relates space, time and gravitation [1]. The theory took time to be fully understood and accepted by the scientific community. In fact, it only began to emerge as a prominent research topic around 1950, mainly due to developements in sectors of physics strictly related with GR. One of these was the astronomical discovery of highly energetic and compact objects, such as quasars and X-ray sources, which seemed to attribute responsibility to the involment of gravitational collapses, singularities and black holes [2]. Another motivation lay in the desire to develop a theory capable of unifying all fundamental interactions. Therefore, a deep understanding of the classical formulation of gravitation was essential. In the standard formulation of GR, the geometrisation of the gravitational interaction is achieved through the curvature of spacetime, which has become the usual interpretation of gravity.

The fundamental pillars of this theory are described by the following principles: the Principle of Relativity, the General Covariance Principle, the Equivalence Principle (EP) and the Principle of Causality. Taken together, they lead to a spacetime structure characterised by two fields: the Lorentzian metric $g$, which determines the causal structure of spacetime, and the linear connection $\Gamma$, which governs the free-fall of bodies.
It is important to remark that $g$ and $\Gamma$ can be *a priori* independent; however, in GR the connection is required to be the Levi-Civita one, built from the metric tensor. Combined with the prescription of parallel transport, these structures encode their physical interpretation in the EP, of which several formulations exist. First of all, the EP states that there always exists a local inertial frame in which gravitational effects can be nullified. Its weak version (WEP) states that the free-fall motion of an uncharged test body is independent of its internal structure and composition, while its strong version (SEP) extends this statement to massive gravitating bodies, treating them as freely falling objects just like test particles. However, even if we commonly relate gravitation interactions to the curvature of the spacetime, it is not necessarily true that this is the best way to represent it. Moreover, the Levi-Civita connection is only a particular choice among many possible connections.

In addition to these technical aspects, although GR has become the widely accepted theory of Gravity and been tested observationally to a very high precision, there still remain fundamental questions to investigate. At Ultraviolet scale, we know that GR is a non-renormalizable theory, meaning that when trying to quantize it, the theory becomes unpredictable at high energies, as it requires an infinite number of counterterms to fix the divergences. Moreover, it is not a gauge theory.
At Infrared scale, there are the open problems of dark matter and dark energy.
Dark matter was originally introduced to explain the observed rotation curves of galaxies, while dark energy was proposed to account for the late-time accelerated expansion of the Universe, as confirmed by observations of Type Ia supernovae [3, 4]. The existence of these two components has been further supported by a wide range of observational evidence, including Baryon Acoustic Oscillations (BAO) [5, 6] and measurements of the Cosmic Microwave Background (CMB) radiation [7]. Nevertheless, their fundamental nature remains elusive, as neither dark matter nor dark energy has been directly detected.

In light of these considerations, and following Einstein's original insight, it is pertinent to explore other possible formulations of Gravity and the equivalent manners in which it can be geometrised. This motivates the study of Metric-Affine Theories of Gravity (MAG). MAG theories are defined by a triplet $\{M, g_{\mu\nu}, \Gamma^{\rho}{}_{\mu\nu}\}$, where $M$ is a four-dimensional spacetime manifold, $g_{\mu\nu}$ is a symmetric rank-two tensor with 10 independent components, and $\Gamma^{\rho}{}_{\mu\nu}$ is an affine connection with 64 independent components. These theories treat the spacetime metric and affine connection as independent variables, unlike GR, which assumes a torsion-free and metric-compatible connection.

In a Metric-Affine framework, the geometry of spacetime may include not only curvature, but also torsion and non-metricity. The knowledge of curvature allows us to measure the rotation experimented by a vector when it is parallel transported along a closed curve, torsion gives a measure of the non-closure of the parallelogram formed when two infinitesimal vectors are parallel transported along each other, while non-metricity measures how much the length of vectors changes as we parallel transport them.

Among MAG theories, we are particularly interested in three special cases that provide dynamically equivalent descriptions of gravity. Specifically, in this work we will consider the standard formulation of GR, based on curvature $R$ and the metric tensor, the Teleparallel Equivalent of General Relativity (TEGR), formulated in terms of torsion $T$ and relying on tetrads and a flat spin connection, and the Symmetric Teleparallel Equivalent of General Relativity (STEGR), based on non-metricity $Q$ and constructed from the metric together with a flat and torsionless affine connection. Unlike GR, the teleparallel formulations arise naturally as gauge theories of translations.

We will show how TEGR and STEGR are constructed and demonstrate that, together with GR, they are dynamically equivalent: their Lagrangians differ only by a boundary term and they lead to the same form of field equations and solutions. For these reasons, GR, TEGR and STEGR form the so-called Geometric Trinity of Gravity. However, these theories also present conceptual differences, such as their relationship with the EP (and especially the SEP) [8]. In GR, the EP is assumed *a priori* as a foundational principle used to identify the correct geometric framework. In TEGR and STEGR, instead, the EP is recovered *a posteriori* (in its strong formulation), suggesting that it may not be a fundamental feature but rather an emergent one, potentially implying differences in the empirical content of the three theories.

Possible methods to detect violations of the EP, WEP and SEP are discussed from both experimental and theoretical perspectives. In particular, by studying the equations of motion of free-falling particles, whose behaviour is crucial when assessing any gravitational theory, we will see that the geodesics do not coincide with autoparallel curves when torsion or non-metricity are present, in contrast with the situation in GR. This indicates that if we consider the equations of motion as an integral part of the description of the theory, then the concept of the Trinity as equivalent description of the same phenomena could be compromised [9]. We will see that these difficulties are closely tied to the nature of the matter under consideration: ambiguities can emerge when matter is coupled to spacetime geometry.

In the final sections, we will briefly discuss these subtleties, which become particularly delicate when considering fermionic fields in the presence of torsion.

# Chapter 1

# Elements of Differential Geometry

In the first part of this chapter, we will provide a collection of concepts and tools of differential geometry that will be useful in the following chapters. In the second part, we will start analyzing some of these notions from a more physical perspective.

## Part I

## 1.1 Differentiable manifolds and bundles

**Definition 1.1.1 (Topological manifold).** A *n-dimensional topological manifold* $M$ is a topological space that satisfies the following requirements:

- Hausdorff, i.e. distinct points have disjoint neighbourhoods;
- second-countable, i.e. $M$ has a countable base;
- locally homeomorphic to the Euclidean space $\mathbb{R}^n$, i.e. for each point $p \in M$, there is an open neighbourhood $\mathcal{U}$ and a homeomorphism $\varphi$ between $\mathcal{U}$ and some open set of $\mathbb{R}^N$.

A pair $(U, \varphi)$ is called *chart*, where $U \subseteq M$ is the chart domain and $\varphi : U \longrightarrow V$ is a homeomorphism with $V \subset \mathbb{R}^n$.
An *atlas* of class $C^k$ on a manifold $M$ is a set $\{(U_\alpha, \varphi_\alpha)\}$ of charts such that the domains $U_\alpha$ cover $M$ and the $\{\varphi_\alpha\}$ satisfy the compatibily condition

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta) \qquad \in C^k$$

**Definition 1.1.2 (Smooth manifold).** A *n-dimensional smooth manifold* is a pair $(M, \mathfrak{D})$ where $M$ is a $n$-dimensional topological manifold and $\mathfrak{D}$ is a $C^\infty$-differentiable structure on it.

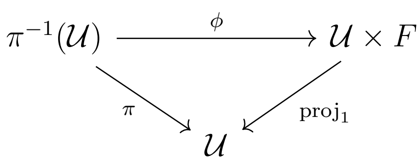
**Definition 1.1.3 (Fiber bundle).** A *fiber bundle*[1] is a set $(E, M, \pi, F)$ where $E$, $M$ and $F$ are topological manifolds and

$$\pi : E \longrightarrow M$$

is a continuous surjective map which satisfies the following condition: for each $p \in M$ there exists an open neighbourhood $\mathcal{U}$ of it and a homeomorphism $\phi : \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times F$, called *local trivialization*, such that

$$\pi = \mathrm{proj}_1 \circ \phi,$$

where $\mathrm{proj}_1 : \mathcal{U} \times F \longrightarrow \mathcal{U}$ is the projection onto the first factor, so that the following diagram commutes:




$E$ is the *total space*, $M$ is the *base manifold*, $F$ is the *(abstract) fiber* and $\pi$ the *projection map* of the fiber bundle. For each point $p \in M$, the set $\pi^{-1}(p)$ is homeomorphic to $F$ and is called the *fiber over* $p$.

The preimage by $\pi$ of any open set $\mathcal{U} \subset M$ is homeomorphic to a product space $\mathcal{U} \times F$, hence $E$ is locally the base manifold $M$ with a "copy" of $F$ attached to each point, but not globally.

A trivial fiber bundle is one that admits a local trivialization over the entire base space (a global trivialization).

**Definition 1.1.4 (Sections on fiber bundles).** Let $(E, M, \pi, F)$ be a fiber bundle. A *local section* around the point $p \in M$ (with $\mathcal{U}$ neighbourhood of $p$) is a continuous map $\sigma : \mathcal{U} \longrightarrow E$ such that $\pi \circ \sigma$ is the identity map.

A *(global) section* of a fiber bundle is a continuous map $\sigma : M \longrightarrow E$ such that $\pi \circ \sigma$ is the identity map.

Intuitively, a section associates to each point of its domain (within the base manifold) an element of the fiber over it, but in a continuous way.

**Example 1.1.1.** Sections of $TM$ are vector fields on $M$.

A very special case of manifolds are **Lie groups**, i.e. groups which carry a manifold structure. With them we can introduce the following concept, very important in gauge theories:

[1]We will make use of the short notation $\pi : E \longrightarrow M$ to denote the fiber bundle.

**Definition 1.1.5 (Principal fiber bundle).** Let $G$ be a Lie group and $M$ a smooth manifold. A *principal fiber bundle* on $M$ with structure group $G$ is given by a manifold $P$ with smooth surjective projection $\pi$ and a right action $\sigma$

$$\pi : P \longrightarrow M, \qquad \sigma : P \times G \longrightarrow P,$$

such that

- $G$ acts freely on $P$, i.e. $\forall p \in P$, $G_p = \{e\}$, which means that the identity element is the only element that remains invariant after the action of $G$ on $M$[2];
- $G$ preserves the fibers $P_x = \pi^{-1}(x)$ of $\pi$, i.e. $\pi(pg) = \pi(p)\ \forall p \in P$ and $g \in G$, and $G$ is transitive on each fiber of $\pi$;
- $\pi : P \longrightarrow M$ is locally trivial, i.e. $\forall x \in M$ exists a neighbourhood $\mathcal{U}$ of $x$ and a diffeomorphism
$$\Psi : P_{\mathcal{U}} = \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times G,$$
which preserves the fibers: $\Psi(p) = (\pi(p), \psi(p))$, with $\psi : P \longrightarrow G$.

## 1.2 Tangent and cotangent spaces

Let $(M, \mathfrak{D})$ be a $n$-dimensional smooth manifold, $p \in M$ and $(U, \varphi = (x^\mu)) \in \mathfrak{D}$ a chart around $p$.

**Definition 1.2.1 (Differentiable function on a manifold).** A map $f : M \longrightarrow \mathbb{R}$ is a *$C^k$-differentiable function over $M$* if $f \circ \varphi^{-1}$ is $C^k$-differentiable.

$C^k(M)$ is the set of all $C^k$-differentiable functions over the manifold.

Let us now focus on $C^\infty(M)$. For a given chart, consider the set of differentiable operators

$$\begin{aligned} \frac{\partial}{\partial x^\mu} : C^\infty(M) &\longrightarrow C^\infty(M) \\ f &\longmapsto \frac{\partial f}{\partial x^\mu}, \end{aligned}$$

so we can define:

[2] $G_p$ is the stabilizer of $G$.

**Definition 1.2.2** (**Tangent space**)**.** The *tangent space* at $p \in M$, namely $T_pM$, is the vector space (isomorphic to $\mathbb{R}^N$) generated by

$$T_pM \coloneqq \operatorname{span}\left\{ \left.\frac{\partial}{\partial x^\mu}\right|_p ,\ \mu = 1, \dots, N \right\}.$$

An arbitrary vector in the tangent space $v = v^\mu \partial_\mu|_p$ ($v^\mu \in \mathbb{R}$) acts on functions $f \in C^\infty(M)$ as $v(f) = v^\mu \partial_\mu f(p)$.

Thus, every tangent vector at a point $p$ can be expressed as a linear combination of the coordinate derivatives $\frac{\partial}{\partial x^\mu}$. The directional derivatives along the coordinate lines at $p$ form a basis of an $n$-dimensional vector space whose elements are the tangent vector at $p$. This space is called tangent space $T_pM$ and the basis $\frac{\partial}{\partial x^\mu}$ is called coordinate basis or **holonomic frame**[3].

By considering the disjoint union of all tangent spaces,

$$TM \coloneqq \bigsqcup_{p \in M} T_pM,$$

we get the *tangent bundle*, whose fibers are the tangent spaces $T_pM = \pi^{-1}(p)$. The underlying bundles structure is $(TM, M, \mathrm{proj}_1, \mathbb{R}^N)$, with total space $TM$.

With the notion of tangent space, we can build the set of *1-forms* on $p$ by duality. The dual basis of $\{\partial_\mu|_p\}$ will be denoted as $\{\mathrm{d}x^\mu|_p\}$, thus the duality relation is $\mathrm{d}x^\mu(\partial_\nu) = \delta^\mu_\nu$. From this follows:

**Definition 1.2.3** (**Cotangent space**)**.** The *cotangent space* at $p \in M$, namely $T_p^*M$, is the dual vector space of the tangent space $T_pM$, i.e.

$$T_p^*M \coloneqq (T_pM)^* = \operatorname{span}\left\{ \mathrm{d}x^\mu|_p ,\ \mu = 1, \dots, N \right\}.$$

The elements of $T_p^*$ are called *covariant vector*, and have *contravariant* indeces.

As before, we can build the *cotangent bundle* as

$$T^*M \coloneqq \bigsqcup_{p \in M} T_p^*M,$$

[3]We will delve into this concept in Sec.(1.9).

with bundle structure $(T^*M, M, \mathrm{proj}_1, \mathbb{R}^N)$.

**Definition 1.2.4 (Smooth bundle).** A bundle $(E, M, \pi, F)$ is called *smooth* if $E$, $M$ and $F$ are smooth manifold and the projection $\pi$ is a $C^\infty$-differentiable map.

**Definition 1.2.5 (Smooth section).** Let $(E, M, \pi, F)$ be a smooth bundle. A section $\sigma : M \longrightarrow E$ (or local) is a *smooth section* if is a $C^\infty$-differentiable map. The set of all smooth sections is $\Gamma(E)$.

## 1.3 Tensor bundles and differential forms

Once built vectors and 1-forms, we can construct more complicated objects by taking the tensor product $\otimes$ of the tangent space and the cotangent space several times:

**Definition 1.3.1 ($(r,s)$-tensor space).** The $(r,s)$-*tensor space* on $T_p^{(r,s)}M$ at $p \in M$ is the vector space given by

$$T_p^{(r,s)}M \coloneqq T_pM \otimes \overset{r \text{ times}}{\cdots} \otimes T_pM \otimes T_p^*M \otimes \overset{s \text{ times}}{\cdots} \otimes T_p^*M.$$

In terms of a particular chart $x^\mu$,

$$T_p^{(r,s)}M \coloneqq \mathrm{span}\Big\{ \partial_{\mu_1}\big|_p \otimes \cdots \otimes \partial_{\mu_r}\big|_p \otimes \mathrm{d}x^{\nu_1}\big|_p \otimes \cdots \otimes \mathrm{d}x^{\nu_s}\big|_p \,, \\ \mu_1, \dots, \mu_r, \nu_1, \dots, \nu_s = 1, \dots, n \Big\}.$$

We say that the tensor space $T_p^{(r,s)}M$ has $r$-contravariant and $s$-covariant tensors on $p$. Once we have a coordinate basis we can drop the tensor space and work directly with the components of the tensor:

$$t = t^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}\, \partial_{\mu_1}\big|_p \otimes \overset{r \text{ times}}{\cdots} \otimes \partial_{\mu_r}\big|_p \otimes \mathrm{d}x^{\nu_1}\big|_p \otimes \overset{s \text{ times}}{\cdots} \otimes \mathrm{d}x^{\nu_s}\big|_p \quad \longrightarrow \quad t^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} \in \mathbb{R}.$$

**Definition 1.3.2 (Alternation map).** The *alternation map* is defined as

$$Alt : \bigotimes^k T^*M \longrightarrow \bigotimes^k T^*M.$$

If we apply the alternation map to a tensor $T$, we obtain:

$$Alt(T)(V_1, \cdots, V_k) \coloneqq \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) T(V_{\sigma(1)}, \dots, V_{\sigma(k)}),$$

where $V_1, \dots, V_k \in \Gamma(M)$ and $\sigma$ belongs to the permutation group with $k$ elements.

**Definition 1.3.3** (**$k$-forms space / $k$-forms on a manifold**). The space of $k$-forms, or *differential forms of rank $k$* at $p \in M$, namely $\Lambda^k_p(T^*M)$, is the subspace of $T_p^{(0,k)}M$ of totally antisymmetric tensors.
If we write the *bundle of $k$-forms* as

$$\begin{aligned}\Lambda^k(T^*M) &:= \bigsqcup_{p\in M} \Lambda^k(T^*_pM) \\ &= \bigcup_{p\in M} \left\{\varphi : T_pM \times \overset{k \text{ times}}{\cdots} \times T_pM \longrightarrow \mathbb{R},\ \varphi \text{ alternating multilinear map}\right\},\end{aligned}$$

then we can define the k-forms $\Omega^k(M)$ as the space of smooth sections of $\Lambda^k(T^*M)$, i.e.

$$\Omega^k(M) = \Gamma(M, \Lambda^k(T^*M)),$$

with

$$\Omega^k(M) = \left\{\sigma : M \longrightarrow \Lambda^k(T^*M) \text{ s.t. } p \circ \sigma = \mathrm{Id}\right\}$$

with $p : \Lambda^k(T^*M) \longrightarrow M$.
It is a vector space and it is infinite-dimensional[4]. In particular, for $k = 0.$ we have $\Lambda^0(T^*M) = M \times \mathbb{R}$ and the space

$$\Omega^0(M) = \{f : M \longrightarrow \mathbb{R} \text{ s.t. } f \text{ is smooth}\}$$

is the set of smooth real functions on $M$, also denoted with $C^\infty(M)$.

**Definition 1.3.4** (**Pullback of a 1-form**). Let $F : M \longrightarrow N$ be a smooth map between smooth manifolds, and $\omega$ a covector field on $N$. Then, the *pullback of $\omega$ by $F$* is the covector field $F^*\omega$ on $M$ defined by:

$$(F^*\omega)_p = dF^*_p(\omega_F(p)).$$

It acts on a vector $v \in T_pM$ by

$$(F^*\omega)_p(v) = \omega_F(p)(dF_p(v)).$$

[4] unless $M$ is a finite set or $k > \dim(M)$

**Definition 1.3.5** (**Wedge product**)**.** Let $T, S$ be two covariant fields with $k$ and $l$ covariant indeces respectively. The *wedge product* of their product is

$$(T \wedge S)(V_1, \dots, V_{k+l}) = \frac{1}{k!l!} \sum_\sigma \text{sgn}(\sigma) T(V_{\sigma(1)}, \dots, V_{\sigma(k)}) S(V_{\sigma(k+1),\dots,V_{\sigma(k+l)}}).$$

In a certain coordinate frame $\{\mathrm{d}x^\mu\}$, it could be written as

$$T \wedge S = \frac{(k+l)!}{k!l!} T_{[\mu_1 \dots \mu_k} S_{\mu_{k+1} \dots \mu_{k+l}]} \mathrm{d}x^{\mu_1} \otimes \dots \otimes \mathrm{d}x^{\mu_{k+l}}.$$

Or equivalently, using differential forms: let $\phi \in \Omega^k$ and $\chi \in \Omega^l$, then

$$\phi \wedge \chi = \frac{(k+l)!}{k!l!} Alt(\phi \otimes \chi).$$

Properties:

- *Multilinearity.* Let $\phi \in \Omega^k(M)$ and $\chi, \psi \in \Omega^l(M)$, then
  - $\phi \wedge (\chi + \psi) = \phi \wedge \chi + \phi \wedge \psi$;
  - $(\phi + \chi) \wedge \psi = \phi \wedge \psi + \chi \wedge \psi$;
  - $(\phi + \chi) \wedge \psi = \phi \wedge \psi + \chi \wedge \psi$;
  - $f(\phi \wedge \chi) = f\phi \wedge \chi + \phi \wedge f\chi$;
  - $\forall f \in C^\infty$, $f\phi|_p = f(p)\phi_{i_1 \dots i_k}(x)\theta^{i_1} \otimes \dots \otimes \theta^{i_k}$, where $\{\theta^i\}$ is a generic basis of the cotangent bundle.
- *Non-commutative.* $\phi \wedge \chi = (-1)^{k \cdot l} \chi \wedge \phi$.
- *Associative.* $(\phi \wedge \chi) \wedge \psi = \phi \wedge (\chi \wedge \psi) = \phi \wedge \chi \wedge \psi$

## 1.4 Diffeomorphisms and field trasformation rules

Let $M_1$ and $M_2$ be two smooth manifold[5] of dimensions $N_1$ and $N_2$, and two arbitrary open subset of them, respcetively, $\mathcal{V}_1$ and $\mathcal{V}_2$.

**Definition 1.4.1** (**Differentiable map**)**.** Consider a map $f : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$. We say that $f$ is a $C^k$*-differentiable map* if for each $p \in \mathcal{V}_1$ there is some coordinate chart $(\mathcal{U}, \varphi)$ around it and a chart $(f(\mathcal{U}), \psi)$ around the image point $f(p)$, such that $\psi \circ f \circ \varphi^{-1} : \mathbb{R}^{N_1} \longrightarrow \mathbb{R}^{N_2}$ is $C^k$.

[5] we omit the differentiable structure of them.

**Definition 1.4.2 (Diffeomorphism).** A homeomorphism $\phi : M_1 \longrightarrow M_2$ is a *diffeomorphism* if $\phi$ and $\phi^{-1}$ are $C^\infty$.

We will be interested in the diffeomorphism $M \longrightarrow M$, which constitute $\mathrm{Diff}(M)$. The pair $(\mathrm{Diff}(M), \circ)$ has the structure of an infinite-dimensional Lie group. The Lie algebra of $\mathrm{Diff}(M)$, namely Lie(Diff$(M)$), consists of all vector fields on $M$ equipped with the Lie bracket of vector fields. This could be seen by making a small change to the coordinate $x$ at each point in space: $x^\mu \mapsto x^\mu + \epsilon h^\mu(x)$, so that the infinitesimal generators are the vector fields

$$L_h = h^\mu(x)\frac{\partial}{\partial x_\mu}.$$

Thus, its algebra is generated by invinitesimal coordinate transformations. Hence, an arbitrary diffeomorphism can be identified with a general coordinate trasformation.

According to how a field transforms under general coordinate transformations, we can define:

- *Scalar fields.* These are the elements of $C^\infty(M)$, i.e. objects that associate a real number to each point of the manifold. They do not change under general coordinate transformations.
- *Vector fields.* They are the sections of the tangent bundle. At each point, the vector field $V$ picks up an element of the tangent space $V(p) = V^\mu(p)\, \partial_\mu\big|_p$, where $V^\mu(p) \in C^\infty$, i.e.

$$\begin{aligned} V : M &\longrightarrow TM \\ p &\longmapsto V^\mu(p)\, \partial_\mu\big|_p \,. \end{aligned}$$

  If we apply a general coordinate transformation $x^\mu \to y^\nu$, the vectors transform as $V'^\nu = \frac{\partial y^\nu}{\partial x^\mu} V^\mu$, since the basis transforms as $\partial_\nu = \frac{\partial x^\mu}{\partial y^\nu}\partial_\mu$.
  We denote the set of smooth vector fields as $\Gamma(TM)$.
- *1-form/covector fields.* They are sections of the cotangent bundle. As for the vector fields, they have a local expression $\omega(p) = \omega_\mu(p)\mathrm{d}x^\mu\big|_p$, where $\alpha_\mu(p) \in C^\infty(M)$, i.e.

$$\begin{aligned} \omega : TM &\longrightarrow \mathbb{R} \\ V(p) &\longmapsto \omega_\mu(p)\mathrm{d}x^\mu\big|_p \,. \end{aligned}$$

  Under a general coordinate transformation $x^\mu \to y^\nu$, 1-forms transform as $\omega'_\nu = \frac{\partial x^\mu}{\partial y^\nu}\omega_\mu$, since the basis transforms as $\mathrm{d}y^\nu = \frac{\partial y^\nu}{\partial x^\mu}\mathrm{d}x^\mu$.
  We denote the set of smooth 1-form fields as $\Omega^1(M)$.

- *(r,s)-tensor fields.* They are sections of the corresponding tensor bundle. Locally they can be expressed as

$$t = t^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_r} \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_s}$$

  and they transforms as

$$t'^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_r}}{\partial x^{\mu_r}} \frac{\partial x^{\nu_1}}{\partial y^{\beta_1}} \cdots \frac{\partial x^{\nu_s}}{\partial y^{\beta_s}} t^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}.$$

We could go on describing other kinds of fields, but they are not useful to this discussion.

**Definition 1.4.3 (Lie bracket).** The *Lie bracket* is defined as

$$[\cdot, \cdot] : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM).$$

Let $V, W \in \Gamma(TM)$ be two vector fields. Their Lie brakcet is thus another field:

$$[V, W] = VW - WV.$$

In components:

$$[V, W]^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu.$$

**Definition 1.4.4 (Curve, velocity, trajectory).** Let $M$ be a smooth manifold, $\mathcal{U} \subseteq M$ an open set and $I$ an interval of the real line. A *smooth curve* on $\mathcal{U}$ is a differentiable function $\gamma : I \longrightarrow \mathcal{U}$. The tangent vector $\dot{\gamma} = u^\mu \partial_\mu$ is called *velocity* of the curve, while the *trajectory* is the image of $\gamma$.

## 1.5 (Pseudo)-Riemannian manifolds

Before introducing a fundamental tool that will allow us to describe gravity in a different framework (see next section), we should retrace our steps and recall what we know so far. We start with:

**Definition 1.5.1 ((Pseudo)-Riemannian manifold).** A *(pseudo)-Riemannian manifold* is a smooth manifold $M$ of dimension $n$ equipped with a 2-covariant tensor field $g$ which defines a non-degenerate quadratic form $g_p(\cdot, \cdot) = T_pM \times T_pM \longrightarrow \mathbb{R}$, with constant signature

$(r, s)$.

Properties:

- $g$ is symmetric, i.e. $g_{ij} = g_{ji}$;
- $\forall p \in M$, the bilinear form $g_p$ is non-degenerate[6], that is, $g_p(v, w) = 0 \, \forall v \in T_pM$, then $w = 0$.

The most interesting cases are:

- $(M, g)$ is called *Riemannian manifold* if the segnature of $g$ is $(n, 0)$. The canonical form of the metric is $(+1, +1, \cdots, +1)$;
- $(M, g)$ is called *pseudo-Riemannian manifold* if the segnature of $g$ is $(r, s)$ with $r \cdot s \neq 0$. The canonical form of the metric is $(\underbrace{-1, \cdots, -1}_{r \text{ times}}, \underbrace{+1, \cdots, +1}_{s \text{ times}})$;
- $(M, g)$ is called *Lorentzian manifold* if the signature of $g$ is $(1, n-1)$. The canonical form of the metric is $(-1, +1, \cdots, +1)$.

Consider a smooth local coordinates on $M$ given by $N$ real-valued functions $(x^1, \cdots, x^n) : U \longrightarrow \mathbb{R}^n$, then the vectors

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

form a basis of $T_pM$, for any $p \in U$.
In the case in which we are considering this basis, the metric tensor acts like: $g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) : U \longrightarrow \mathbb{R}$, where $U$ is the domain of the coordinate chart.
In terms of tensor algebra, the metric tensor could be written in terms of the dual basis $\mathrm{d}x^1, \cdots, \mathrm{d}x^n$ of the cotangent bundle as

$$g = g_{\mu\nu} \, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu.$$

Due to the non-degeneracy of $g$, each metric (scalar product in the tangent space) has an inverse (scalar product in the cotangent bundle), i.e. a 2-contravariant symmetric and non-degenerate tensor field

$$g^{-1} = g^{\mu\nu} \, \partial_\mu \otimes \partial_\nu$$

[6]Thus, it is diagonalizable and has only non-zero eigenvalues.

such that

$$g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu.$$

Furthermore, the metric defines an isomorphism, the *canonical/musical isomorphism*, between $T_pM$ and $T_p^*M$, at each point $p \in M$. This isomorphism could be generated only if $M$ is equipped with a metric. They are:

$$\begin{aligned} \flat : T_pM &\longrightarrow T_p^*M \\ V = V^\mu \partial_\mu\big|_p &\longmapsto V^\flat = V^\nu g_{\mu\nu} \mathrm{d}x^\mu\Big|_p \equiv V_\mu \mathrm{d}x^\mu\big|_p \end{aligned}$$

$$\begin{aligned} \sharp : T_p^*M &\longrightarrow T_pM \\ \omega = \omega_\mu \mathrm{d}x^\mu\big|_p &\longmapsto \omega^\sharp = \omega_\nu g^{\mu\nu} \partial_\mu\Big|_p \equiv \omega^\mu \partial_\mu\big|_p . \end{aligned}$$

## 1.6 Connections on vector bundles

**Definition 1.6.1 (Vector valued differential forms).** Let $\pi : E \longrightarrow M$ be a real vector bundle.
A *differential $k$-form on $M$ with values in $E$* is a collection of alternating $k$-forms

$$\omega_p : T_pM \times \cdots \times T_pM \longrightarrow E_p,$$

for each $p \in M$, such that the map

$$\begin{aligned} \sigma : M &\longrightarrow E \\ p &\longmapsto \omega_p(X_1(p), \ldots, X_k(p)) \end{aligned}$$

is a smooth section of $E$ for every $k$ vector fields $X_1, \ldots, X_k \in \Gamma(TM)$.
The space of the $k$-forms on $M$ with values in $E$ is denoted by $\Omega^k(M, E) = \Omega^k(E)$.
This type of $k$-form could also be defined as a section of the vector bundle $\Lambda^k(T^*M) \otimes E \longrightarrow M$, thus

$$\Omega^k(M, E) = \Gamma(M, \Lambda^k(T^*M) \otimes E) = \Omega^0(M, \Lambda^k(T^*M) \otimes E).$$

**Remark 1.6.1.** Equivalently, we can define this space as

$$\Omega^k(M, E) = \Gamma(E) \otimes_{C^\infty(M)} \Omega^k(M),$$

hence the elements of $\Omega^k(M, E)$ are linear combinations of $v \otimes \omega$, where $s \in \Gamma(E)$ and $\omega \in \Omega^k(M)$.

A connection on a fiber bundle is a consistent way to move from one fiber in the tangent bundle to another.

**Definition 1.6.2 (Connection on vector bundles).** Let $\pi : E \longrightarrow M$ be a fiber bundle. Then a *connection* on $E$ is a linear map

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$$

such that $\forall f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$, the Leibinz rule holds:

$$\nabla(f\sigma) = \mathrm{d}f \otimes \sigma + f\nabla\sigma.$$

The connection $\nabla$ induces a covariant derivative along vector fields on sections on $E$.

**Definition 1.6.3 (Covariant derivative).** Let $\nabla$ be a connection on the vector bundle $E$ over $M$. Let $X \in \Gamma(TM)$. Then the map

$$\begin{array}{rcl} \nabla_X : \Gamma(E) & \longrightarrow & \Gamma(E) \\ \sigma & \longmapsto & \nabla\sigma(X) \end{array}$$

is called *covariant derivative* of $\sigma$ along $X$ (with respect to $\nabla$).

Properties:

- $f \in C^\infty(M), \sigma \in \Gamma(E)$, then

$$\nabla_X(f\sigma) = \mathrm{d}f(X)\sigma + f\nabla_X\sigma = X(f)\sigma + f\nabla_X\sigma.$$

- $\nabla_X\sigma$ is $C^\infty(M)$-linear in $X$, i.e. $\forall f, g \in C^\infty(M), \forall X_1, X_2 \in \Gamma(TM)$, then

$$\nabla_{fX_1+gX_2}s = f\nabla_{X_1}\sigma + g\nabla_{X_2}\sigma.$$

- $\forall p \in M, (\nabla_X s)(p) \in E_p$ depends only on $X_p \in T_pM$, i.e.

$$\text{if } X_1, X_2 \in \Gamma(TM) \text{ and } X_1(p) = X_2(p) \implies (\nabla_{X_1}\sigma)(p) = (\nabla_{X_2}\sigma)(p).$$

**Theorem 1.6.1.** Each fiber bundle $\pi : E \longrightarrow M$, with $M$ compact, admits a connection.

Once defined the covariant derivative, we can specialize it to some cases of interest.

**Definition 1.6.4 (Flat connection).** In the case of the trivial bundle $E = M \times \mathbb{R}^k$, the sections $\underline{s} \in \Gamma(E)$ are smooth maps $\underline{s} = (s_1, \ldots, s_k)$ with $s_j \in C^\infty(M)$. The flat connection acts on $\underline{s}$ taking the exterior derivative component by component:

$$\nabla \underline{s} = (\mathrm{d}s_1, \ldots, \mathrm{d}s_k) =: \mathrm{d}\underline{s}.$$

The result is a 1-form with value in $\mathbb{R}^k$, i.e. an element of $\Gamma(T^*M \otimes \mathbb{R}^k)$.

**Definition 1.6.5 (Levi-Civita connection).** Let $(M, g)$ a Riemannian manifold, with $g$ metric on $TM$.
The *Levi-Civita connection* $\nabla^{LC}$ is

$$\nabla^{LC} : \Gamma(TM) \longrightarrow \Omega^1(TM),$$

and satisfies the following properties:

- *metric compatibility.* $\forall X, Y, Z \in \Gamma(TM)$,

$$\nabla_Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \tag{1.6.1}$$

- *torsion free.* $\forall X, Y, Z \in \Gamma(TM)$

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{1.6.2}$$

After introducing the Levi-Civita connection, it is useful to describe its local expression in coordinates. For this purpose, we introduce the *Christoffel symbols.*

**Definition 1.6.6 (Christoffel symbols).** Let $(U, \{x^i\})$ be a local coordinate chart on $M$, and let $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$ be the associated local frame of vector fields. The connection $\nabla$ acts on vector fields as follows:

$$\nabla_{\frac{\partial}{\partial x^i}} \left( \frac{\partial}{\partial x^j} \right) = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x^k}, \tag{1.6.3}$$

where the coefficients $\Gamma_{ij}^k$ are called the *Christoffel symbols.*

In the case of the Levi-Civita connection, these symbols can be expressed in terms of the metric $g$ as:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right). \tag{1.6.4}$$

**Theorem 1.6.2 (Levi-Civita).** In a Riemmannian manifold, the Levi-Civita connection always exists and it is unique. Moreover, it is the only one which is torsion free and metric preserver.

**Remark 1.6.2.** If $\nabla$ is a connection on the fiber bundle $\pi : E \longrightarrow M$ and there exists an open $U \subseteq M$, then $\nabla$ induces a connection over $E|_U$.

### 1.6.1 Spin structure on vector bundles

If we want to introduce the concept of spin connection, we have to recall some important facts about spin geometry [10].
First of all, it is covenient recalling what a Clifford algebra is.
Let $(V, (\cdot,\cdot))$ a vectorial space on $\mathbb{R}$ equipped with a symmetric bilinear form

$$\begin{aligned} (\cdot,\cdot) : V \times V &\longrightarrow \mathbb{R} \\ v, v &\longmapsto q(v) = (v,v) \;\; \forall v \in V. \end{aligned}$$

**Definition 1.6.7 (The Clifford algebra).** The *Clifford algebra* $Cl(V,q)$ of $\mathbb{R}$ is the associative algebra on $\mathbb{R}$ given by

$$Cl(V,q) \coloneqq T(V) \big/ (v \otimes v = q(v)),$$

where

$$T(V) = V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots = \bigoplus_k V^{\otimes k}$$

is the tensorial algebra on $V$.
Thus, the Clifford algebra is the largest associative algebra generated by $V$, subject to the relation $v^2 = (v,v) = q(v)$ for all $v \in V$.

Given the real vector space $(V,q)$, we denote with $(V_{\mathbb{C}}, q_{\mathbb{C}})$ its complexification, where $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ and $q_{\mathbb{C}}$ the complex extension of $q$. Then, $Cl(V_{\mathbb{C}}, q_{\mathbb{C}}) \cong Cl(V,q)_{\mathbb{C}} = Cl(V,q) \otimes \mathbb{C}$.

Choose now $V = \mathbb{R}^n$ with the Euclidean norm, namely $(\mathbb{R}^n, \|\cdot\|)$, and let $Cl^{\mathbb{C}}(\mathbb{R}^n)$ be the complexified algebra of $Cl(\mathbb{R}^n)$.

**Definition 1.6.8 (Spin representation of $Cl^{\mathbb{C}}(\mathbb{R}^n)$).** The *spin representation* of $Cl^{\mathbb{C}}(\mathbb{R}^n)$,

$$\tilde{k} : Cl^{\mathbb{C}}(\mathbb{R}^n) \longrightarrow U(\Delta_n),$$

is the unique irreducible representation of $Cl^{\mathbb{C}}(\mathbb{R}^n)$ of dimension $2^n$.
$U(\Delta_n) = \mathrm{Aut}(\Delta_n)$ denotes the group of unitary operators acting on the Hilbert space of spinors $\Delta_n$.

Given the spinor representations of the complex Clifford algebra $\tilde{k}$, we can now define the Spin representations of the spin group $k$, knowing that it is possible to restrict the representation of $Cl^{\mathbb{C}}(\mathbb{R})^n$ to the spin group:

**Definition 1.6.9 (Complex spin representation of $\mathrm{Spin}_n$).** Let $\mathrm{Spin}_n \subset Cl(\mathbb{R}^n) \subset Cl^{\mathbb{C}}(\mathbb{R}^n)$. Then the *complex spin representation* of $\mathrm{Spin}_n$ is the restriction of the complex spin representation of $Cl^{\mathbb{C}}(\mathbb{R}^n)$, namely

$$k : \mathrm{Spin}_n \longrightarrow U(\Delta_n).$$

**Definition 1.6.10 (Spin structure).** Let $M$ be a paracompact[7] manifold and $E$ an oriented vector bundle on $M$ of dimension $n$ equipped with an inner product.
The collection of oriented orthonormal frames of a vector bundle form a frame bundle[8] $P_{SO}(E)$, which is a principal bundle under the action of $SO(n)$.
A *spin stucture* for $P_{SO}(E)$ is a bundle homomorphism [11] over $M$ of principal bundles between $P_{SO}(E)$ and $P_{\mathrm{spin}}(E)$ under the action of the spin group $\mathrm{Spin}_n$, i.e. there exists a map

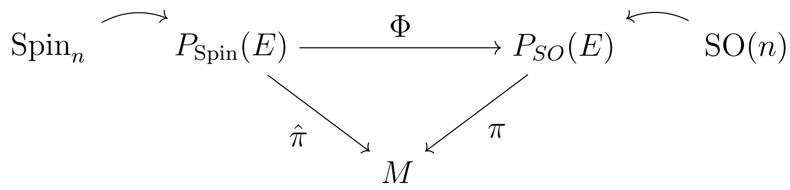
$$\Phi : P_{\mathrm{spin}}(E) \longrightarrow P_{SO}(E)$$

such that

$$\Phi(pg) = \Phi(p)\rho(g), \;\; \pi \circ \Phi = \hat{\pi} \qquad \forall p \in P_{\mathrm{spin}}(E), \forall g \in \mathrm{Spin}_n,$$

where $\rho : \mathrm{Spin}_n \longrightarrow SO(n)$, which means that the spin group is the double cover of $SO(n)$.

[7] Every open cover admits a locally finite subcover. This assures that a metric over $M$ could always be defined.
[8] See Def.(1.9.2) and (1.9.3).




A spinor bundle of $E$ is a prescription that associates a spin representation to every point of $M$.

**Definition 1.6.11 (Spinor bundle).** The *spinor bundle* is defined to be the complex vector bundle

$$S = P_{\mathrm{spin}} \times_k \Delta_n,$$

associated to the spin structure $P_{\mathrm{spin}}$ via the complex spinor representation $k : \mathrm{Spin}_n \longrightarrow U(\Delta_n)$.

The spin representation $k$ is a faithful and unitary representation of the group $\mathrm{Spin}_n$. The spinor bundle is the associated bundle of the principal bundle $P_{\mathrm{spin}}$ through the spin representation.

A section $\psi \in \Gamma(S)$ of the spinor bundle $S$ is called *spinor field* and is a map

$$\begin{aligned} \psi : M &\longrightarrow S \\ x &\longmapsto \psi(x) \in S_x \cong \Delta_n. \end{aligned}$$

Thus, at each point $x \in M$, the fiber $S_x$ is a complex vector space isomorphic to the spinor space $\Delta_n$.

Finally, we can define what a spin connection is. This will be essential for a complete understanding of the following chapters.

Let $\Omega$ be a connection on the oriented vector bundle $E$. It induces a connection $\nabla$ on the $SO(n)$-bundle of oriented orthonormal frames, namely $P_{SO}(E)$, described by 1-forms $\omega_\alpha \in \Omega^1(U_\alpha, SO(n))$.

**Definition 1.6.12 (Spin connection).** The *spin connection* $\tilde{\nabla}$ on the spinor bundle $S$ is defined by the 1-forms

$$\tilde{\omega}_\alpha = \rho_*^{-1}(\omega_\alpha) \in \Omega^1(U_\alpha, \mathrm{Aut}(\Delta_n)), \tag{1.6.5}$$

where $\rho : \mathrm{Spin}_n \longrightarrow SO(n)$.
Hence, the spin connection is defined via the pull-back of the connection defined on the $P_{SO}(E)$.

For the moment, we will limit ourselves to this general definition. Later, after the introduction of the physical aspects too, we will give a more suitable definition for our case of interest.

## 1.7 Torsion, curvature and non-metricity tensor

With the introduction of the covariant derivative, we can construct some fundamental tensors useful to describe the geometry of the manifold. Actually, we have implicitly made use of some of them in the definition of the Levi-Civita connection (1.6.5), in fact conditions (1.6.1) and (1.6.2) can be written by the introduction of two tensors, the **non-metricity tensor** and the **torsion tensor**, respectively.
It is important to note that the following definitions hold in general, meaning that the affine connection $\Gamma$ does not necessarily have to be the Levi-Civita's.

Let $X, Y, Z \in \Gamma(TM)$, then we define these tensors and their respective components in a coordinate basis:

- *Torsion tensor*

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{1.7.1}$$
$$T^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\rho\nu} - \Gamma^{\mu}{}_{\nu\rho}; \tag{1.7.2}$$

- *Curvature tensor*

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.7.3}$$
$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\alpha\rho}\Gamma^{\alpha}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\alpha\sigma}\Gamma^{\alpha}{}_{\nu\rho}; \tag{1.7.4}$$

- *Non-metricity tensor*

$$Q(X,Y,Z) = \nabla_Z(g(X,Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y), \tag{1.7.5}$$
$$Q_{\mu\nu\rho} = \nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^{\lambda}{}_{\mu\nu} g_{\lambda\rho} - \Gamma^{\lambda}{}_{\mu\rho} g_{\nu\lambda}. \tag{1.7.6}$$

These three tensors have the following properties, that hold in general:

$$T^{\mu}{}_{\nu\rho} = -T^{\mu}{}_{\rho\nu}; \tag{1.7.7}$$

$$R^{\mu}{}_{\nu\rho\sigma} = -R^{\mu}{}_{\nu\sigma\rho}; \tag{1.7.8}$$

$$Q_{\mu\nu\rho} = Q_{\mu\rho\nu}. \tag{1.7.9}$$

Hence in the case of the Levi-Civita connection, $T(X,Y) = 0$ and $Q(X,Y,Z) = 0$. We will comprehend the importance of these objects in the next chapters (see Chap.(3)).

### 1.7.1 Bianchi's identities

The characterizing tensors of a general affine connection are not completely independent from each other. In particular, curvature and torsion are related by the *Bianchi's identities*:

$$R^{\mu}{}_{[\nu\rho\sigma]} = \nabla_{[\nu}T^{\mu}{}_{\rho\sigma]} + T^{\mu}{}_{\alpha[\nu}T^{\alpha}{}_{\rho\sigma]}; \tag{1.7.10}$$

$$\nabla_{[\alpha}R^{\mu}{}_{|\nu|\rho\sigma]} = -R^{\mu}{}_{\nu\tau[\alpha}T^{\tau}{}_{\rho\sigma]}. \tag{1.7.11}$$

## 1.8 Parallel transport of tangent vectors

In Euclidean space, tangent spaces at different points can be naturally identified through translation, making it straightforward to transfer vectors between them. However, on a general Riemannian manifold, moving tangent vectors from one point to another along a curve requires a process called *parallel transport*. This process depends on a chosen affine connection. Given a specific affine connection, there is a unique way to perform parallel transport of tangent vectors. Different affine connections, however, will yield different parallel transport behaviors.

**Definition 1.8.1.** Let $\gamma : [a,b] \subseteq \mathbb{R} \longrightarrow M$ a smooth curve. The derivative of a vector field $X \in \Gamma(TM)$ along a curve $\gamma$ with tangent vector $\dot{\gamma}(\lambda) = (\mathrm{d}\gamma^{\mu}/\mathrm{d}\lambda)\partial_{\mu}$, is defined as

$$\frac{DX}{\mathrm{d}\lambda} = \nabla_{\dot{\gamma}}X. \tag{1.8.1}$$

Then, it is possible to define a parallel transport for $X$ associated with this specific connection. $X$ is said to be *parallel transported* along $\gamma$ if

$$\frac{DX}{\mathrm{d}\lambda} = 0 \iff \nabla_{\dot{\gamma}}X = 0 \qquad \forall\lambda \in [a,b], \tag{1.8.2}$$

namely,

$$\frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda} + \Gamma^\mu{}_{\rho\nu}\frac{\mathrm{d}\gamma^\rho}{\mathrm{d}\lambda}X^\nu = 0. \tag{1.8.3}$$

Hence, the covariant derivative of the field $X$ along the tangent curve $\dot{\gamma}$ is zero at each point of the curve. Eq.(1.8.3) represents a system of first order differential equations in the unknown $X^\mu$, which admits a unique solution once the initial condition $X_0^\mu = X^\mu(\lambda_0)$ has been provided. It is important to note that $X^\mu(\gamma(\lambda))$ depends on the curve $\gamma$.

**Theorem 1.8.1.** For any curve $\gamma : [a, b] \subseteq \mathbb{R} \longrightarrow M$, any $\lambda_0 \in [a, b]$ and any $X_0 \in T_{\gamma(\lambda_0)}M$, there exists an unique vector field $X$ along $\gamma$ which is parallel, such that $X(\gamma(\lambda_0)) = X_0$.

By evaluating a parallel vector field at two points $x$ and $y$, one obtains an identification between a tangent vector at $x$ and a tangent vector at $y$. In this context, the tangent vector at $y$ is considered the *parallel transport* of the tangent vector at $x$ along the given path. Vectors related in this way are said to be parallel transports of each other.

**Definition 1.8.2 (Parallel transport).** The *parallel transport* of a vector field $X$ from $\gamma(\lambda_0)$ to $\gamma(\lambda)$ along $\gamma$ is denoted by

$$\begin{aligned} P^\gamma_{\lambda_0,\lambda} : T_{\gamma(\lambda_0)}M &\longrightarrow T_{\gamma(\lambda)}M \\ X_0 = X(\gamma(0)) &\longmapsto X(\gamma(\lambda)) \end{aligned}$$

and it is a linear isomorphism.

**Definition 1.8.3 (Holonomy group).** Let $\nabla$ be a connection on the tangent bundle $TM$, and let $x \in M$. For any smooth loop $\gamma : [0, 1] \to M$ based at $x$, i.e. $\gamma(0) = \gamma(1) = x$, the parallel transport along $\gamma$ defines a linear isomorphism

$$P^\gamma := P^\gamma_{0,1} : T_xM \longrightarrow T_xM.$$

The set of all such linear maps forms a subgroup of the general linear group $GL(T_xM)$, called the *holonomy group* of $\nabla$ at $x$:

$$\mathrm{Hol}_x(\nabla) = \{\, P^\gamma \in GL(T_xM) \text{ s.t. } \gamma \text{ is a smooth loop based at } x \,\}.$$

**Definition 1.8.4 (Autoparallel).** Consider a general connection $\Gamma^\rho{}_{\mu\nu}$. An *autoparallel* is a curve $\gamma(\lambda)$ whose velocity $\dot{\gamma} = (\mathrm{d}\gamma^\mu/\mathrm{d}\lambda)\partial_\mu$ is invariant under parallel transport up to a term proportional to the velocity, i.e.

$$\nabla_{\dot{\gamma}}\gamma = \frac{\mathrm{d}^2\gamma^\rho}{\mathrm{d}\lambda^2} + \Gamma^\rho{}_{\mu\nu}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}\gamma^\nu}{\mathrm{d}\lambda} = f(\lambda)\gamma^\rho. \tag{1.8.4}$$

If $f(\lambda) = 0$, then the autoparallel is said to be *affinely parametrized*.
Eq.(1.8.4) is a system of second order differential equations, which admit a unique solution once initial position and velocity have been assigned.

**Definition 1.8.5 (Geodesic).** A *geodesic* is a particular case of autopalell, in which the general connection is constrained to be the Levi-Civita one. Hence, the affinely parametrized geodesic is given by

$$\overset{\circ}{\nabla}_{\dot{\gamma}}\gamma = \frac{\mathrm{d}^2\gamma^\rho}{\mathrm{d}\lambda^2} + \overset{\circ}{\Gamma}{}^\rho{}_{\mu\nu}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}\gamma^\nu}{\mathrm{d}\lambda} = 0, \tag{1.8.5}$$

where we named $\nabla^{LC} = \overset{\circ}{\nabla}$ for future convenience.
We will return on the differences between geodesics and autoparallels in the next chapters.

### 1.8.1 Geodesic deviation equation

The geodesic deviation equation is the equation that relates the tendency of geodesics to accelerate toward or away from each other to the curvature of the manifold. This gives another characterization of curvature, and it also plays an important role in motivaring Einstein's equations (see Sec.(2.2)).

Let $\gamma_s(t)$ denote a smooth one-parameter family of goedesics, hence for each $s \in \mathbb{R}$, the curve $\gamma_s$ is a geodesic (parametrized by the affine parameter $t$), and the map $(t, s) \longmapsto \gamma_s(t)$ is smooth, one-to-one and has smooth inverse. Let $\Sigma$ denote the two-dimensional submanifold spanned by the curves $\gamma_s(t)$.
This family of geodesics forms a congruence on $\Sigma$, which means that each point of $\Sigma$ belongs to the image of one and only one curve of the family. For this reason, the pair $(t, s)$ may be choosen as the coordinates of $\Sigma$.
A natural vector basis adapted to the coordinate system is given by the pair $\{T^\mu, S^\mu\}$, where $T^\mu = \frac{\partial x^\mu}{\partial t}$ is tangent to the family of geodesics and thus satisfies

$$T^\mu \overset{\circ}{\nabla}_\mu T^\nu = 0, \tag{1.8.6}$$

while the vector field $S^\mu = \frac{\partial x^\mu}{\partial s}$ represent the displacement to an infinitesimally nearby geodesic, and it is called *deviation vector* [12].

We can define

$$V^\mu = T^\nu \mathring{\nabla}_\nu S^\mu \tag{1.8.7}$$

as the quantity that gives the rate of change along a geodesic of the displacement to an infinitesimally nearby geodesic, and it is called *relative velocity*. Moreover, we can define the quantity

$$A^\mu = T^\nu \mathring{\nabla}_\nu V^\mu \tag{1.8.8}$$

as the *relative acceleration* of an infinitesimally nearby geodesic in the family.
With these ingredients, it is possible to derive an equation which relates $A^\mu$ to the Riemann tensor:

$$\begin{aligned} A^\mu &= T^\nu \mathring{\nabla}_\nu \big(T^\lambda \mathring{\nabla}_\lambda S^\mu\big) \\ &= T^\nu \mathring{\nabla}_\nu \big(S^\lambda \mathring{\nabla}_\lambda T^\mu\big) \\ &= T^\nu \big(\mathring{\nabla}_\nu S^\lambda\big) \mathring{\nabla}_\lambda T^\mu + T^\nu S^\lambda \mathring{\nabla}_\nu \mathring{\nabla}_\lambda T^\mu \\ &= S^\nu \mathring{\nabla}_\nu T^\lambda \mathring{\nabla}_\lambda T^\mu + T^\nu S^\lambda \big(\mathring{\nabla}_\lambda \mathring{\nabla}_\nu T^\mu + \mathring{R}^\mu{}_{\sigma\nu\lambda} T^\sigma\big) \\ &= S^\nu \mathring{\nabla}_\nu T^\lambda \mathring{\nabla}_\lambda T^\mu + T^\lambda S^\nu \mathring{\nabla}_\nu \mathring{\nabla}_\lambda T^\mu + T^\lambda S^\nu \mathring{R}^\mu{}_{\sigma\lambda\nu} T^\sigma \\ &= S^\nu \mathring{\nabla}_\nu \big(T^\lambda \mathring{\nabla}_\lambda T^\mu\big) + \mathring{R}^\mu{}_{\sigma\lambda\nu} T^\sigma \\ &= \mathring{R}^\mu{}_{\sigma\lambda\nu} T^\sigma T^\lambda S^\nu \end{aligned} \tag{1.8.9}$$

where in the second line we use the fact that, since the torsion is null, then $[S, T] = [\partial_s, \partial_t] = 0$ and hence $T^\lambda \mathring{\nabla}_\lambda S^\mu = S^\lambda \mathring{\nabla}_\lambda T^\mu$. Instead, in the fourth line we use the definition of Riemann tensor, namely $[\mathring{\nabla}_\nu, \mathring{\nabla}_\lambda]T^\mu = R^\mu{}_{\sigma\nu\lambda} T^\sigma$ (see eq.(1.7.3)).
Eq.(1.8.9) is the *geodesic deviation equation*, from which follows that the Riemann tensor is non-zero if and only if the geodesics accelerate towards eachother.
Initially parallel geodesics, i.e. $V^\mu = 0$, remain parallel if and only if the Riemann tensor vanishes.

## 1.9 Frames and coframes. Anholonomy.

So far we used a formalism in which the basis of the tangent space are given by coordinate systems. In this chapter, we want to develop a framework in which we consider any sort of basis, with some properties. In this framework, the nexus with gauge theories is clearer. We will deepler explain this and its physical interpretation in Sec.(1.10).

**Definition 1.9.1 (Frame over a manifold).** Let $\pi : E \longrightarrow M$ be a vector bundle, with $U \subseteq M$ an open subset. A $k$-tuple of local sections $(\sigma_1, \dots, \sigma_k)$ of $E$ over $U$ is said to be linearly independent if their values $(\sigma_1(p), \dots, \sigma_k(p))$ form a linear independent $k$-tuple in $E_p$ for each $p \in U$.
$(\sigma_1(p), \dots, \sigma_k(p))$ spans $E$ if their values span $E_p$ for each $p \in U$.
A *local frame* for $E$ over $U$ is an ordered $k$-tuple $(\sigma_1, \dots, \sigma_k)$ of linear independent local sections over $U$ that span $E$. Thus, $(\sigma_1(p), \dots, \sigma_k(p))$ is a basis for the fiber $E_p$ for each $p \in U$. It is called global frame if $U = M$.
We denote the set of all frames at $p$ as $P_p$.

**Remark 1.9.1.** $P_p$ has a natural right action by the $GL(n, \mathbb{R})$ group: if $g \in GL(n, \mathbb{R})$, it acts on the frame $fr$ via composition to give the new frame:

$$fr \circ g : \mathbb{R}^n \longrightarrow E_p$$

such that the action on $P_p$ is both free and transitive.

**Definition 1.9.2 (Frame bundle).** A *frame bundle* $P(E)$ of $E$ is the disjoint union of all the $P_p$:

$$P(E) := \bigsqcup_{p \in M} P_p.$$

Each point in $P(E)$ is a pair $(p, fr)$ where $x \in M$ and $fr$ is a frame at $p$.

**Definition 1.9.3 (Orthonormal frame).** A local (or global) frame consisting of orthonormal vector fields with respect to inner product equipped on the manifold is called *orthonormal frame*.

A particular case of frame bundle is:

**Definition 1.9.4 (Tangent frame bundle).** The *tangent frame bundle* $LM$ of a smooth manifold $M$ is the frame bundle associated with the tangent bundle $TM$ of $M$.
If $M$ is $n$-dimensional then the tangent bundle has rank $n$, and the frame bundle of $M$ is a principal $GL(n, \mathbb{R})$-bundle over $M$.

Analogously, we can define the *(linear) coframe bundle* $L^*M$ by taking all the basis of the contangent bundle at each point. We represent frames with $\{e_a\}$ and coframes with $\{\theta^a\}$.

**Definition 1.9.5 (Parallelizable manifold).** Let $M$ be a smooth manifold of dim$M = N$. $M$ is said to be *parallelizable* if it admits a global frame (global basis field) $\{X_1, \dots, X_N\}$ of $M$ defined over all $TM$ such that for each $p \in M$, $\{X_1(p), \dots, X_N(p)\}$ is a basis of $T_pM$, where $T_pM$ denotes the fiber over $p$ of the tangert bundle $TM$.

**Remark 1.9.2.** Another way to define a parallelizable manifold is by saying that it is so if and only if there exists a global section $\sigma$.

**Example 1.9.1.** Lie groups are parallelizable manifolds, because since their algebra is (well) defined by all left-invariant fields, it is always possible to build a global section $\sigma \in C^\infty$ that associates to each point a basis of the tangent space.

By remembering what we said in Def.(1.2.2),

**Definition 1.9.6 (Holonomic frame).** A (local) frame $\{X_1, \dots, X_n\}$ is *holonomic* if there exixts a coordinate system $\{x^1, \dots, x^n\}$ such that $X_i = \frac{\partial}{\partial x^i}$.

**Remark 1.9.3.** A preferred class of frames is that of *inertial frames*, denoted with $e'_a$, whose coefficients of anholonomy $f'^c{}_{ab}$ locally satisfy the condition $f'^c{}_{ab} = 0$. Thus, these frames are holonomic, and all the coordinate frames belong to this family.

**Theorem 1.9.1.** A local frame $\{X^1, \dots, X^n\}$ is holonomic if and only if $[E_i, E_j] = 0$, i.e. the frame vector fields commute.

Hence, not every basis field is a coordinate system. In fact, even if a manifold is parallelizable, it may not admit a global coordinate frame. On the contrary, if it admits a global coordinate frame, it implies that the manifold is homeomorphic to the euclidean space, because the exsitence a global coordinate frame, so there is a global homeomorphism $\Psi : M \longrightarrow \mathbb{R}^n$.

Therefore, instead of using $e_\mu = \partial_\mu$ and $\theta^\mu = \mathrm{d}x^\mu$ as basis of $T_pM$ and $T_p^*M$ respectively, we can define a set of vectors and covectors, which we call[9] $e_a$ and $\theta^a$ with $a = 0, 1, 2, \dots$, and we choose them to be orthonormal, i.e.

$$g(e_a, e_b) = \eta_{ab}, \tag{1.9.1}$$

[9] We will use Greek indeces $(\mu, \nu, \rho, \dots)$ for coordinate frames and Latin indeces $(a, b, c, \dots)$ for arbitrary frames.

with $\eta$ the Minkowsi metric.
Now we can expand $e_a$ in the coordinate base $e_\mu = \partial_\mu$:

$$e_a = e_a{}^\mu e_\mu, \tag{1.9.2}$$

where the set of coefficients $\{e_a{}^\mu\}$ are called *matrix of tetrad tranformation* and belong to the linear group of all real $4\times 4$ invertible matrices $\mathrm{GL}(4,\mathbb{R})$: the new basis $\{e_a\}$ is the frame of basis vectors which is obtained by a rotation of the holonomic basis $\{\partial_\mu\}$, preserving the orientation.
The inverse matrix is $e^a{}_\mu$ with

$$e^a{}_\mu e_b{}^\mu = \delta^a{}_b \quad \text{and} \quad e_a{}^\mu e^a{}_\nu = \delta^\mu{}_\nu. \tag{1.9.3}$$

With this writing, eq.(1.9.1) becomes

$$g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \eta_{ab}, \tag{1.9.4}$$

or, equivalently

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}. \tag{1.9.5}$$

**Remark 1.9.4.** $g$ is $e^2$ up to $\eta$, in fact $e$ is called *metric square*.

**Remark 1.9.5.** If we switch gravitation off, the metric tensor $g_{\mu\nu}$ is replaced by the Minkowski metric $\eta_{\mu\nu}$, hence the relation (1.9.4) is now

$$\eta_{\mu\nu} e_a{}^\mu e_b{}^\nu = \eta_{ab}. \tag{1.9.6}$$

These are called *trivial frames*, represent inertial frames and account only for inertial effects (with no contribution from spacetime curvature), as it is a geodesic motion.

By means of the same procedure, we can choose the basis in the cotangent space to be $\theta^a$, and we choose it such that the duality condition is satisfied:

$$\theta^a(e_b) = \delta^a{}_b. \tag{1.9.7}$$

This equation implies

$$\theta^\mu = e_a{}^\mu \theta^a \quad \text{and} \quad \theta^a = e^a{}_\mu \theta^\mu.$$

*Proof.* $\theta^a(e_b) = \delta^a{}_b \implies \theta^a{}_\mu \theta^\mu(e_b{}^\nu e_\nu) = \delta^a{}_b \implies \theta^a{}_\mu e_b{}^\nu \underbrace{\theta^\mu(e_\nu)}_{\delta^\mu{}_\nu} = \delta^a{}_b \implies \theta^a{}_\mu e_b{}^\nu = \delta^a{}_b$

$$\implies \theta^a{}_\mu = e^a{}_\mu$$

□

**Remark 1.9.6.** We can view the tetrad as an object that transforms Greek indeces in Latin indeces, and viceversa.

Thus, at the end, in terms of coordinate frame, we have obtained:

$$\boxed{e_a = e_a{}^\mu \partial_\mu, \qquad e^a = e^a{}_\mu \mathrm{d}x^\mu} \tag{1.9.8}$$

where we have denote $\theta^a = e^a$ with an abuse of notation.

**Example 1.9.2.** To formally understand the notion of a tetrad, consider a two-dimensional manifold (see Fig.(1.1)). In this setting, the coordinate basis $(\partial_x, \partial_y)$ of the tangent space at a point is, by construction, tangent to the coordinate lines but not necessarily orthonormal with respect to the underlying metric. In contrast, a tetrad (or vielbein) provides an orthonormal basis $(e_1, e_2)$, where each vector is expressed as a linear combination of the coordinate basis vectors:

$$e_1 = e_1{}^\mu \partial_\mu, \quad e_2 = e_2{}^\mu \partial_\mu.$$

**Remark 1.9.7.** A coordinate frame $\{\partial_\mu\}$ is related to coordinates. A general frames $\{e_a\}$ does not necessarily have this property. If it has, then there must exist certain functions $y^a(x)$ (the new coordinates) such that

$$e_a{}^\mu = \frac{\partial x^\mu}{\partial y^a}, \qquad \text{(integrability condition)}$$

that will not happen.

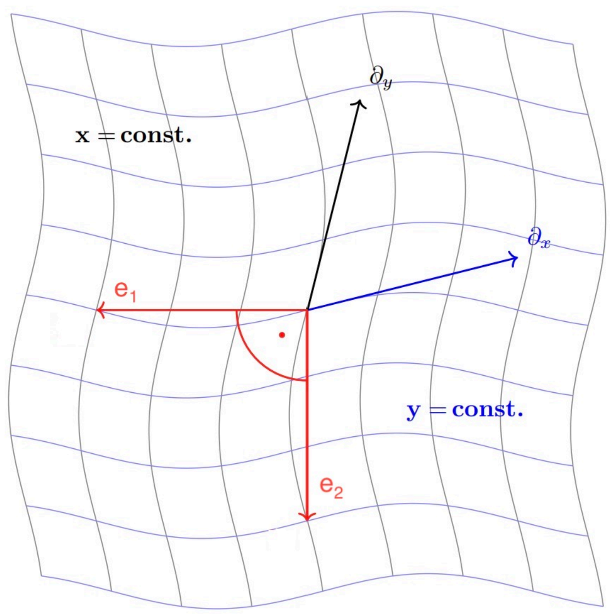



FIGURE 1.1: visualization of a coordinate basis and tetrad basis in a two-dimensional case. Figure credits [13].

We can understand this better: the Lie bracket (1.4.3) of two tetrad fields is:

$$
\begin{aligned}
[e_a, e_b] &= e_a e_b - e_b e_a = (e_a{}^{\mu}\partial_{\mu})(e_b{}^{\nu}\partial_{\nu}) - (e_b{}^{\nu}\partial_{\nu})(e_a{}^{\mu}\partial_{\mu}) \\
&= (e_a{}^{\mu}\partial_{\mu})(e_b{}^{\nu}e^c{}_{\nu}e_c) - (e_b{}^{\nu}\partial_{\nu})(e_a{}^{\mu}e^c{}_{\mu}e_c) \\
&= e_a{}^{\mu}[(\partial_{\mu}e_b{}^{\nu})e^c{}_{\nu}e_c + e_b{}^{\nu}(\partial_{\mu}e^c{}_{\nu})e_c + e_b{}^{\nu}e^c{}_{\nu}\partial_{\nu}e_c]+ \\
&\qquad - e_b{}^{\nu}[(\partial_{\nu}e_a{}^{\mu})e^c{}_{\mu}e_c + e_a{}^{\mu}(\partial_{\nu}e^c{}_{\mu})e_c + e_a{}^{\mu}e^c{}_{\mu}\partial_{\nu}e_c] \\
&= e_a{}^{\mu}e_b{}^{\nu}[(\partial_{\mu}e_b{}^{\nu})e^c{}_{\nu}e_c + (\partial_{\mu}e^c{}_{\nu})e_c + e^c{}_{\nu}\partial_{\mu}e_c]+ \\
&\qquad - e_b{}^{\nu}e_a{}^{\mu}[(\partial_{\nu}e_a{}^{\mu})e^c{}_{\mu}e_c + (\partial_{\nu}e^c{}_{\mu})e_c + e^c{}_{\mu}\partial_{\nu}e_c] \\
&= e_a{}^{\mu}e_b{}^{\nu}[\cancel{(\partial_{\mu}e_b{}^{\nu})e^c{}_{\nu}e_c} + (\partial_{\mu}e^c{}_{\nu})e_c + \cancel{e^c{}_{\nu}\partial_{\mu}e_c}]+ \\
&\qquad - e_b{}^{\nu}e_a{}^{\mu}[\cancel{(\partial_{\nu}e_a{}^{\mu})e^c{}_{\mu}e_c} + (\partial_{\nu}e^c{}_{\mu})e_c + \cancel{e^c{}_{\mu}\partial_{\nu}e_c}] \\
&= e_a{}^{\mu}e_b{}^{\nu}(\partial_{\mu}e^c{}_{\nu} - \partial_{\nu}e^c{}_{\mu})e_c \\
&= f^c{}_{ab}e_c.
\end{aligned}
$$

In the fifth line we collected $e_b{}^\nu$, and $e_a{}^\mu$ in the seventh, while the cancellation is due the following trick:

$$(1.9.1) \implies e^a{}_\nu = \delta^\mu{}_\nu e^a{}_\mu \implies e^c{}_\mu \partial_\nu e_c = \delta^\nu{}_\mu e^c{}_\nu \partial_\nu e_c = e^c{}_\nu \partial_\mu e_c.$$

The same goes for the other pair. Thus:

$$\implies \boxed{[e_a, e_b] = f^c{}_{ab} e_c} \tag{1.9.9}$$

where

$$\boxed{f^c{}_{ab} = e_a{}^\mu e_b{}^\nu (\partial_\mu e^c{}_\nu - \partial_\nu e^c{}_\mu)} \tag{1.9.10}$$

are known as *anholonomy coefficients*. They quantify the failure of parallelogram closurness generated by the vectors $e_a$ and $e_b$. So we can find out if a frame is associated to coordinates or not, and thus, state the following theorem:

**Theorem 1.9.2.** The basis $\{e_a\}$ is a coordinate frame, i.e. it is integrable, if and only if the anholonomy coefficients vanish: $f^c{}_{ab} = 0$.

Now it is clear why previuosly we have defined coordinate frames as holonomic frames: because they have zero anholonomy.
In general, when $f^c{}_{ab} \neq 0$, the tetrad basis is anholonomic and the anholonomy coefficients specify how much they depart from being holonomic.

# Part II - The need for a spin connection

Up to this point, we have introduced several concepts from a purely mathematical perspective. We now aim to incorporate physical insights into the discussion. In particular, we will define the spin connection more physically, explore its significance, and understand its role within the broader physical context.

## 1.10 Lorentz invariance and coupling prescription

Lorentz invariance is a fundamental symmetry of nature. Any relativistic equation could be written in a Lorentz covariant form.
Tetrad fields $\{e^a{}_\mu\}$ represent observers in special relativity, and allow the projection of vectors and tensors of the spacetime in the local frame of an observer (locally flat spacetime). To measure field quantities with both magnitude and direction, an observer must project them onto their own reference frame. Spin connections play a very important role because they represent the inertial effects occurring in the considered frame. It is called spin connection because it could be used to build covariant derivatives for spinors to describe a fermion in a curved background (coupling with gravity).

To introduce the concept of the spin connection (also called Lorentz connection), let us think about what happens in a quantum field theory [14] when we want to make a theory, for instance a Dirac theory with

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi,$$

invariant under a local $U(1)$ phase transformation

$$\psi(x) \to e^{i\alpha(x)}\psi(x),$$

which goes under the name of Abelian gauge field theory. Since under local $U(1)$ each point of the representation space of the Dirac field transforms in a different way, the usefulness of a directional derivative is now missing. If we want to construct a derivative, i.e. being able to compare fields in different spacetime points, we have to parallel transport a field on the tangent space of the other one.
This leads to the introduction of an additional term in the derivative,

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu, \tag{1.10.1}$$

where $A_\mu$ is called gauge field, acts linearly on the field and encodes the local infinitesimal transformation that we have done. $D_\mu$ is called covariant derivative and the additional term allows the Dirac Lagrangian to remain unchanged under a $U(1)$ transformation.

Mathematically, $A_\mu$ is a connection, which is a 1-form with value in the Lie algebra of the Lie group $U(1)$. In the case discussed (as in many others, e.g. Chromodynamics), however, the connection related to the gauge transformation takes value in groups that do not directly involve the spacetime.

Problems arise when describing certain theories of gravity. In fact, as we will see, it is possible to construct different gauge formulations of gravity, in which the General Relativity is formulated as a gauge theory of translations. Then, it is clear that in this process we must take in account the spacetime, because this theory works with spacetime itself.
Under a change of frame, a tensor transforms according to representations of $GL(4, \mathbb{R})$, which is the set of all invertible four-dimensional real matrices, but some of these objects may lose their covariant behaviour under local (point-dependent) transformations. In order to re-establish the covariance, we have to introduce connections, called spin connections, (which depend on the case we are considering) that allow us to compare vectors at different points via parallel transport in a way that is consistent with the spacetime structure and therefore has to take values in $\mathfrak{gl}(4, \mathbb{R})$. To be more precise, as we will see in the next section, the structure group from which the connection derives its values will be restricted from $GL(4, \mathbb{R})$ to $SO^+(3, 1)$.
This operation produces a modification of the derivative, similar to (1.10.1) and is called *coupling prescription*. It is for this reason that **the spin connection can be regarded as the gauge field generated by local Lorentz transformations**.

Before going any further, we would like to recall the most important aspects of the Lorentz groups and its algebra.

## 1.11 The Lorentz group

The Lorentz group is a group of linear coordinate transformations applied on four-vectors. We indicate such transformation with $\Lambda$:

$$x^\mu \longmapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1.11.1}$$

such that the square of four-vector in Minkowski spacetime remains invariant, i.e.

$$x^\mu x_\mu = \eta_{\mu\nu} x^\mu x^\nu \implies \eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} (\Lambda^\mu{}_\rho x^\rho)(\Lambda^\nu{}_\sigma x^\sigma) = \eta_{\rho\sigma} x^\rho x^\sigma$$
$$\implies \boxed{\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}} \tag{1.11.2}$$

The inverse of $\Lambda$ is $\Lambda^{-1}$ and it is defined as $(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu$. As $\Lambda^{-1}$ is a Lorentz transformation too, eq.(1.11.2) still holds:

$$\eta_{\mu\nu} (\Lambda^{-1})^\mu{}_\rho (\Lambda^{-1})^\nu{}_\sigma = \eta_{\rho\sigma} \implies \eta_{\mu\nu} \Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu = \eta_{\rho\sigma}$$
$$\implies \eta^{\mu\nu} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = \eta^{\rho\sigma}$$

In matrix notation: $\eta = \Lambda^T \eta \Lambda \implies (\det\Lambda)^2 = \mathrm{Id}$. This means $\det\Lambda = \pm 1$.

- $\det\Lambda = +1$ encodes transformation called *proper* Lorentz transformation. The product of two proper Lorentz transformation is a proper transformation, hence they form subgroup called $SO(3,1)$.
- $\det\Lambda = -1$ encodes transformation called *improper* Lorentz transformation. They can be written as product of a proper Lorentz transformation and a discrete transformation that changes sign of odd number of coordinates.

Consider now eq.(1.11.2) with $\rho = \sigma = 0$: $\eta_{00} = 1 = \eta_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = (\Lambda^0{}_0)^2 - \sum_{i=1}^3 (\Lambda^i{}_0)^2$.

$$\implies \Lambda^0{}_0 \geq 1 \text{ or } \Lambda^0{}_0 \leq -1.$$

Lorentz group with $\Lambda^0{}_0 \geq 1$ produces *orthocronus* transformations, while $\Lambda^0{}_0 \leq -1$ produces *non-orthocronus* transformations.
We are interested in proper orthocronus Lorentz transformations, which forms the *restricted Lorentz special orthogonal group* $SO^+(3,1)$.

**Remark 1.11.1.** $SO^+(3,1)$ is a Lie group and $SO^+(3,1) \subset GL(4,\mathbb{R})$.

The expansion near the identity is $\Lambda = \mathrm{Id} + \omega + o(\omega^2)$, i.e.

$$\Lambda^\mu{}_\nu = \eta^\mu{}_\nu + \omega^\mu{}_\nu + o(\omega^2). \tag{1.11.3}$$

If we insert eq.(1.11.3) into eq.(1.11.2), we obtain

$$\begin{aligned}(\eta^{\nu}{}_{\nu} + \omega^{\mu}{}_{\nu} + o(\omega^2))(\eta^{\rho}{}_{\sigma} + \omega^{\rho}{}_{\sigma} + o(\omega^2)) &= \eta^{\mu}{}_{\nu}\eta^{\rho}{}_{\sigma}\eta^{\nu\sigma} + \eta^{\mu}{}_{\nu}\omega^{\rho}{}_{\sigma}\eta^{\nu\sigma} + \omega^{\mu}{}_{\nu}\eta^{\rho}{}_{\sigma}\eta^{\nu\sigma} + o(\omega^2) \\ &= \eta^{\nu\rho} + \omega^{\rho\mu} + \omega^{\mu\rho} + o(\omega^2) \stackrel{!}{=} \eta^{\mu\rho} \\ &\implies \omega^{\rho\mu} = -\omega^{\mu\rho}. \end{aligned} \tag{1.11.4}$$

$\omega$ are generic elements of $\mathfrak{so}(3,1)$, and the most generic $4 \times 4$ antisymmetric matrix has six independent components: three of these are related to boost transformations and the other three to angular momentum transformations (rotation in $\mathbb{R}^3$).

### 1.11.1 Representations of the Lorentz group

A representation $R$ of a group $G$ is an operation that associates a linear operator $D_R$ with every element $g \in G$ such that the group structure is preserved:

$$D_R(g_1, g_2) = D_R(g_1)D_R(g_2).$$

For a representation of a Lie group with parameters $\theta^a$, we have

$$D_R(g(\theta)) = e^{i\theta^a T^a_R},$$

where $T^a_R$ are the generators of the group in the representation $R$. For an infinitesimal $\theta^a$, $D_R(\theta) \simeq 1 + i\theta^a T^a_R$. Generators form a Lie algebra: $[T^a, T^b] = if^{ab}{}_c T^c$, where $f^{ab}{}_c$ are the structure coefficients, independent of the representation $R$.

For the Lorentz algebra, the parameters are $\omega^{\mu\nu}$, then

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}, \tag{1.11.5}$$

where $J^{\mu\nu}$ are the generators while the $\frac{1}{2}$ takes into account the fact that both $\omega_{\mu\nu}$ and $J^{\mu\nu}$ are antisymmetric.

## 1.12 Local transformations of tetrad bases

A set of tetrad fields is a collection of four orthonormal, linearly independent vector fields in spacetime $\{e_1{}^{\mu}, e_2{}^{\mu}, e_3{}^{\mu}, e_4{}^{\mu}\}$. They constitute the local reference frame of an observer that moves along a trajectory $\gamma$, represented by the worldline $x^\mu(\tau)$, where $\tau$ is the proper time of the observer [15].

The components ${e_0}^\mu$ and $\{{e_i}^\mu\}$ are timelike and spacelike vectors, respectively[10]. The set $\{{e_a}^\mu\}$ transforms as a contravariant vector field under coordinate transformations of the spacetime, and as a covariant vector field under $SO^+(3,1)$.
The basis $e_a$ could be changed indipendendly from the coordinates:

$$e_a \longmapsto e'_a = {\Lambda_a}^b(x) e_b. \tag{1.12.1}$$

In each tangent space we can choose a different $\Lambda$, hence these are transformations that depend on the point.
We want the new basis $e'_a$ to be orthonormal as well, thus $g(e'_a, e'_b) = \eta_{ab}$ must hold.

$$\begin{aligned} \implies g({\Lambda_a}^c e_c, {\Lambda_b}^d e_d) = \eta_{ab} &\implies {\Lambda_a}^c {\Lambda_b}^d g(e_c, e_d) = \eta_{ab} \\ &\implies {\Lambda_a}^c {\Lambda_b}^d \eta_{cd} = \eta_{ab}. \end{aligned} \tag{1.12.2}$$

This means that the transformations ${\Lambda_a}^b(x)$ have to leave the metric unchanged at each point: they are analogous to the Lorentz transformations of the Special Relativity, but in this case they are local, thus they can be different depending on the considered tangent space.

To summarize, so far we encountered two gauges: General Relativity is invariant under change of coordinates (diffeomorphisms) and there is also a freedom to do local Lorentz transformations. If we make such transformations together, a tensor with both Latin and Greek indeces transforms as

$$\boxed{{X^{c\sigma}}_{d\lambda} = {\Lambda^c}_a \frac{\partial x'^\sigma}{\partial x^\mu} {(\Lambda^{-1})^b}_d \frac{\partial x^\nu}{\partial x'^\lambda} {X^{a\mu}}_{b\nu}} \tag{1.12.3}$$

The covariant derivative of a tensor with only Greek indeces contains corrections given by the connection coefficients ${\Gamma^\alpha}_{\beta\gamma}$, needed to make the tensor transform as a tensor under diffeomorphisms, thus they deal with spacetime/external indeces.
The same procedure could be done for an orthonormal basis, with the connection coefficients replaced with the spin coefficients ${\omega^{ab}}_\mu$, which deal with the tangent space/internal indeces.

[10] $g_{\mu\nu} {e_0}^\mu {e_0}^\nu = \eta_{00} = -1$, i.e timelike. Same goes for $\{{e_i}^\mu\}$

## 1.13 The spin connection and the Fock-Ivanenko derivative

Now, we have all the ingredients to specify the definition of spin connection (1.6.12) given before.
Let us recall that we are now working with $SO^+(3,1)$ as structure group of the gauge invariance, since we said that the spin conncetion could be interpreted as a gauge potential generated by local Lorentz transformations.

The connection that belongs to the principal $SO^+(3,1)$-bundle is

$$\omega \in \Omega^1(P_{SO}, \mathfrak{so}(3,1)), \tag{1.13.1}$$

hence takes value in the $\mathfrak{so}(3,1)$ algebra. In this particular case, the spin connection is the pull-back of $\omega$ via

$$\rho : \mathrm{Spin}(3,1) \longrightarrow SO(3,1), \tag{1.13.2}$$

namely, $\tilde{\omega} = \rho_*^{-1}(\omega)$. Thus, $\tilde{\omega} \in \Omega^1(P_{\text{spin}}, \mathfrak{spin}(3,1))$.
Note that $\mathrm{Spin}(3,1) \cong SL(2,\mathbb{C})$ double covers $SO(3,1)$. However, the spinorial representation is a representation of $\mathfrak{so}(3,1)$ via the generators $\Sigma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b]$. This means that to construct the spin connection is sufficient $\mathfrak{so}(3,1)$, in fact it is commonly said that **the spin connection is a $\mathfrak{so}(3,1)$-valued 1-form on the principal $P_{\text{Spin}}$-bundle.**

**Remark 1.13.1.** The frame bundle of the spacetime is a principal $SO^+(3,1)$-bundle. This means that at each spacetime point, the set of all possible frames for the tangent space forms the fiber: the transitive action of the structure group $SO^+(3,1)$ on the fibers allows the construction of the entire fiber by shifting a point via Lorentz transformation.

Physically, we write the spin connection as linear combination of the $\mathfrak{so}(3,1)$ Lie algebra generators:

$$\omega_\mu = \frac{1}{2}\omega^{ab}{}_\mu J_{ab} \quad \in C^\infty(M) \otimes \mathfrak{so}(3,1), \tag{1.13.3}$$

where $\omega^{ab}{}_\mu$ are the spin connection coefficients, which are antisymmetric in the latin indeces, that is $\omega^{ab}{}_\mu = -\omega^{ba}{}_\mu$, due to the antisymmetricity of $J^{ab}$.

**Remark 1.13.2.** $\omega^{ab} = \omega^{ab}{}_\mu \mathrm{d}x^\mu \in \Omega^1(M)$.

**Remark 1.13.3.** We can offer an alternative perspective on the restriction from the group $GL(4,\mathbb{R})$ to $SO^+(3,\mathbb{R})$. Spinors transform under a specific representation of the Lorentz algebra $\mathfrak{so}(3,1)$, but this representation does not extend to $\mathfrak{gl}(4,\mathbb{R})$, in which the Levi-Civita connection takes values. As a result, the Levi-Civita connection cannot naturally act on spinors. To define a covariant derivative for spinors, we require a connection that is compatible with their transformation properties. This necessitates reducing the structure group to obtain a $\mathfrak{so}(3,1)$-valued connection that can act appropriately on spinors.

This definition allows us to introduce the *Fock-Ivanenko derivative*:

$$\boxed{\mathcal{D}_\mu := \partial_\mu - \omega_\mu = \partial_\mu - \frac{i}{2}\omega^{ab}{}_\mu J_{ab},} \tag{1.13.4}$$

where $J_{ab}$ are the generators of the appropriate representation of the Lorentz group.

**Example 1.13.1.** If we want to describe the Dirac equation $(i\partial\!\!\!/ - m)\psi = 0$ in a curved background, we have to replace $\partial\!\!\!/$ with $\mathcal{D}\!\!\!/ = \gamma^\mu(\partial_\mu + \frac{1}{4}\omega^{ab}{}_\mu\Sigma_{ab})$ where $\Sigma_{ab} = [\gamma_a, \gamma_b]$ are the generator of $\mathfrak{spin}(3,1)$.

Using the four vector representation of the Lorentz algebra generators, namely

$$(J_{ab})^c{}_d = i(\eta_{bd}\delta^c{}_a - \eta_{ad}\delta^c{}_b), \tag{1.13.5}$$

we can compute the Fock-Ivanenko derivative of a tetrad field:

$$\begin{aligned}
\mathcal{D}_\mu e^c &= (\partial_\mu - \omega_\mu)e^c = \partial_\mu - \frac{i}{2}\omega^{ab}{}_\mu[i(\eta_{bd}\delta^c{}_a - \eta_{ad}\delta^c{}_b)]e^d \\
&= \partial_\mu e^c + \frac{1}{2}[\omega^a{}_{d\mu}\delta^c{}_a - \omega_d{}^b{}_\mu\delta^c{}_b]e^d = \partial_\mu e^c + \frac{1}{2}[\omega^c{}_{d\mu} - \omega_d{}^c{}_\mu]e^d \\
&= \partial_\mu e^c + \frac{1}{2}(2\omega^a{}_{d\mu})e^d \qquad \text{(using the antisymmetry of } \omega^{ab}{}_\mu) \\
&= \partial_\mu e^c + \omega^c{}_{d\mu}e^d
\end{aligned} \tag{1.13.6}$$

From this, we can find the Fock-Ivanenko derivative of a tetrad matrix by using eq.(1.9.8), namely $e^a = e^a{}_\mu \mathrm{d}x^\mu$:

$$\boxed{\mathcal{D}_\mu(e^c{}_\lambda) = \partial_\mu(e^c{}_\lambda) + \omega^c{}_{d\mu}e^d{}_\lambda} \tag{1.13.7}$$

## 1.14 Relation between $\Gamma^{\gamma}{}_{\mu\nu}$ and $\omega^{ab}{}_{\mu}$: the tedrad postulate

We saw that the Fock-Ivanenko derivative acts on internal indeces and can be defined for both tensorial and spinorial fields. In fact, more generically, the covariant derivative of $\tilde{\nabla}$ of a tensor $X^a{}_b$ is:

$$\tilde{\nabla}_\mu X^a{}_b = \partial_\mu X^a{}_b + \omega^a{}_{c\mu} X^c{}_b - \omega^c{}_{b\mu} X^a{}_c, \tag{1.14.1}$$

where each Latin index brings a interaction term which contains a spin correction.
$\tilde{\nabla}_\mu X^a{}_b$ transforms eq.(1.12.3) in the right way under diffeomorphisms and local Lorentz transformations. The covariant derivative of a field $X$ is

$$\begin{aligned} \nabla X &= (\nabla_\mu X^\nu) \mathrm{d}x^\mu \otimes \partial_\nu \\ &= (\partial_\mu X^\nu + \Gamma^\nu{}_{\mu\lambda} X^\lambda) \mathrm{d}x^\mu \otimes \partial_\nu . \end{aligned} \tag{1.14.2}$$

On the other hand, we can write the same covariant derivative but in a mixed basis:

$$\begin{aligned} \tilde{\nabla} X &= (\nabla_\mu X^a) \mathrm{d}x^\mu \otimes e_a = (\partial_\mu X^a + \omega^a{}_{b\mu} X^b) \mathrm{d}x^\mu \otimes e_a \\ &= [\partial_\mu (e^a{}_\nu X^\nu) + \omega^a{}_{b\mu} e^b{}_\lambda X^\lambda] \mathrm{d}x^\mu \otimes (e_a{}^\sigma \partial_\sigma) \\ &= e_a{}^\sigma [e^a{}_\nu \partial_\mu X^\nu + X^\nu \partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b e^b{}_\lambda X^\lambda] \mathrm{d}x^\mu \otimes \partial_\sigma \\ &= [\partial_\mu X^\sigma + e_a{}^\sigma X^\nu \partial_\mu e^a{}_\nu + e_a{}^\sigma e^b{}_\lambda \omega^a{}_{b\mu} X^\lambda] \mathrm{d}x^\mu \otimes \partial_\sigma . \end{aligned} \tag{1.14.3}$$

Since the covariant derivative of a field cannot depend on the basis, then $\nabla = \tilde{\nabla}$ (tetrad postulate), hence eq.(1.14.2) and eq.(1.14.3) must correspond. For this purpose, let us rename the dummy indeces $\sigma \mapsto \nu$ and $\nu \mapsto \lambda$ of eq.(1.14.3) and compare it with eq.(1.14.2):

$$\Gamma^\nu{}_{\mu\lambda} = e_a{}^\nu \partial_\mu e^a{}_\lambda + e_a{}^\nu e^b{}_\lambda \omega^a{}_{b\mu} = e_a{}^\nu \mathcal{D}_\mu e^a{}_\lambda, \tag{1.14.4}$$

which could be written as:

$$\begin{aligned} \omega^a{}_{b\mu} &= e^a{}_\nu e_b{}^\lambda \Gamma^\nu{}_{\mu\lambda} - e_b{}^\lambda \partial_\mu e^a{}_\lambda \\ &= e^a{}_\nu e_b{}^\lambda \Gamma^\nu{}_{\mu\lambda} + e^a{}_\lambda \partial_\mu e_b{}^\lambda \\ &= e^a{}_\nu \nabla_\mu e_b{}^\nu , \end{aligned} \tag{1.14.5}$$

where in the second line we use the fact that $\partial_\mu(e_b{}^\lambda e^a{}_\lambda) = \partial_\mu(\delta^a{}_b) = 0$, hence $e^a{}_\lambda \partial_\mu e_b{}^\lambda + e_b{}^\lambda \partial_\mu e^a{}_\lambda = 0$.

**Remark 1.14.1.** The spin connection is then canonically induced by affine connection, thus is entirely dependent on the metric. The spin connection is, in a sort of sense, a lift of the affine connection from the tangent bundle.

Eq.(1.14.5) is of a remarkable importance, because it is equivalent to

$$\boxed{\nabla_\mu e^a{}_\nu = 0} \tag{1.14.6}$$

which means that the tetrad is *covariantly constant* under both diffeomorphisms and local Lorentz transformations.

*Proof.*

$$\begin{aligned} \nabla_\mu e^a{}_\nu = 0 &\Longleftrightarrow \partial_\mu e^a{}_\nu - \Gamma^\rho{}_{\mu\nu} e^a{}_\rho + \omega^a{}_{b\mu} e^b{}_\nu = 0 \\ &\Longleftrightarrow e_c{}^\nu \partial_\mu e^a{}_\nu - e_c{}^\nu \Gamma^\rho{}_{\mu\nu} e^a{}_\rho + \omega^a{}_{b\mu} \underbrace{e^b{}_\nu e_c{}^\nu}_{\delta^b{}_c} = 0 \qquad \text{(contracting with } e_c{}^\nu) \\ &\Longleftrightarrow \omega^a{}_{c\mu} = e_c{}^\nu \Gamma^\rho{}_{\mu\nu} e^a{}_\rho - e_c{}^\nu \partial_\mu e^a{}_\nu \end{aligned}$$

□

**Remark 1.14.2.** Another way to say that the metric is covariantly constant, namely $\nabla_\lambda g_{\mu\nu} = 0$, is by saying that the spin connection is antisymmetric in the Lorentz indeces, meaning that it takes value in the Lorentz algebra. Thus, *the metric compatibility holds if and only if we choose a Lorentz connection.*

*Proof.*

$$\begin{aligned} 0 = \nabla_\lambda g_{\mu\nu} &= \partial_\lambda g_{\mu\nu} - \Gamma^\sigma{}_{\lambda\mu} g_{\sigma\nu} - \Gamma^\sigma{}_{\lambda\nu} g_{\mu\sigma} \\ &= \underbrace{\partial_\lambda (e^a{}_\mu e^b{}_\nu \eta_{ab})}_{=0} - e_a{}^\sigma g_{\sigma\nu} \mathcal{D}_\lambda e^a{}_\mu - e_a{}^\sigma g_{\mu\sigma} \mathcal{D}_\lambda e^a{}_\nu \qquad \text{(using (1.9.4) and (1.14.4))} \\ &= e_a{}^\sigma g_{\sigma\nu} (\partial_\lambda e^a{}_\mu + \omega^a{}_{d\lambda} e^d{}_\mu) + e_a{}^\sigma g_{\mu\sigma} (\partial_\lambda e^a{}_\nu + \omega^a{}_{d\lambda} e^d{}_\nu) \qquad \text{(using (1.13.7))} \\ &= e^b{}_\nu (g_{\sigma\nu} e_a{}^\sigma e_b{}^\nu \omega^a{}_{d\lambda} e^d{}_\mu + e_a{}^\sigma e_b{}^\nu g_{\mu\sigma} \omega^a{}_{d\lambda} e^d{}_\nu) \\ &= e^b{}_\nu e^d{}_\mu (\eta_{ab} \omega^a{}_{d\lambda} + e_a{}^\sigma \eta_{ad} \omega^a{}_{b\lambda}) \\ &= e^b{}_\nu e^d{}_\mu (\omega_{bd\lambda} + \omega_{db\lambda}) \\ &\Longrightarrow \omega_{bd\lambda} = -\omega_{db\lambda} \end{aligned}$$

□

Hence, at the end we need to remember these two fundamental results:

$$\nabla_\mu e^a{}_\nu = \partial_\mu e^a{}_\nu - \Gamma^\rho{}_{\mu\nu} e^a{}_\rho + \omega^a{}_{b\mu} e^b{}_\nu = 0, \qquad \text{(tetrad postulate)} \tag{1.14.7}$$

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\sigma{}_{\lambda\mu} g_{\sigma\nu} - \Gamma^\sigma{}_{\lambda\nu} g_{\mu\sigma} = 0. \qquad \text{(metric compatibility)} \tag{1.14.8}$$

## 1.15 The dynamical variables in tetrad formalism

Having the definitions of tetrad and spin connection, it is possibile to compute torsion, curvature and non-metricity in this formalism. An important advantage of formulating gravity using tetrads and a spin connection - especially in the context of Teleparallel geometries - is that these definitions depend solely on the spin connection. As a result, whether these quantities vanish or not becomes a property of the spin connection alone and remains unaffected by the choice of tetrad. Making use of Defs.(1.7.1), (1.7.3) and (1.7.5), and of a coordinate basis $\{\partial_\mu\}$ as $\{e_\mu\}$ we obtain:

$$\begin{aligned}
T^a{}_{\mu\nu} &= \langle e^a, T(e_\mu, e_\nu)\rangle \\
&= \langle e^a, \nabla_{e_\mu} e_\nu - \nabla_{e_\nu} e_\mu - \overbrace{[e_\mu, e_\nu]}^{=0}\rangle \\
&= \langle e^a{}_\rho dx^\rho, (\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}) e_\lambda\rangle && \text{(using } \nabla_{e_\mu} e_\nu = \Gamma^\lambda{}_{\mu\nu} e_\lambda) \\
&= e^a{}_\lambda (\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) \\
&= \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu} e^b{}_\nu - \omega^a{}_{b\nu} e^b{}_\mu. && \text{(using eq.(1.14.6))}
\end{aligned} \tag{1.15.1}$$

$$
\begin{aligned}
R^a{}_{b\mu\nu} &= \langle e^a, R(e_\mu, e_\nu, e_b)\rangle \\
&= \langle e^a, (\nabla_{e_\mu}\nabla_{e_\nu} - \nabla_{e_\nu}\nabla_{e_\mu} - \overbrace{\nabla_{[e_\mu,e_\nu]}}^{=0})e_b{}^\rho e_\rho\rangle \\
&= \langle e^a{}_\sigma dx^\sigma, \nabla_{e_\mu}[(\partial_\nu e_b{}^\rho)e_\rho + e_b{}^\rho \nabla_{e_\nu} e_\rho] - \nabla_{e_\nu}[(\partial_\mu e_b{}^\rho)e_\rho + e_b{}^\rho\nabla_{e_\mu} e_\rho]\rangle \\
&= \langle e^a{}_\sigma dx^\sigma, \cancel{(\partial_\mu\partial_\nu e_b{}^\rho)e_\rho} + (\partial_\nu e_b{}^\rho)\nabla_{e_\mu} e_\rho + \cancel{(\partial_\mu e_b{}^\rho)\nabla_{e_\nu} e_\rho} + e_b{}^\rho \nabla_{e_\mu}\nabla_{e_\nu} e_\rho \\
&\qquad - \cancel{(\partial_\nu\partial_\mu e_b{}^\rho)e_\rho} - (\partial_\mu e_b{}^\rho)\nabla_{e_\nu} e_\rho - \cancel{(\partial_\nu e_b{}^\rho)\nabla_{e_\mu} e_\rho} - e_b{}^\rho\nabla_{e_\nu}\nabla_{e_\mu} e_\rho\rangle \\
&= \langle e^a{}_\sigma dx^\sigma, (\partial_\nu e_b{}^\rho)\nabla_{e_\mu} e_\rho - (\partial_\mu e_b{}^\rho)\nabla_{e_\nu} e_\rho + e_b{}^\rho(\nabla_{e_\mu}\nabla_{e_\nu} - \nabla_{e_\nu}\nabla_{e_\mu})e_\rho\rangle \\
&= \langle e^a{}_\sigma dx^\sigma, (\Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu})\partial_\lambda(e_b{}^\rho)e_\rho\rangle + e^a{}_\sigma e_b{}^\rho R^\sigma{}_{\rho\mu\nu} \\
&= (\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu})e_b{}^\rho\partial_\lambda e^a{}_\rho + e^a{}_\sigma e_b{}^\rho R^\sigma{}_{\rho\mu\nu} \\
&= \partial_\mu\omega^a{}_{b\nu} - \partial_\nu\omega^a{}_{b\mu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu}.
\end{aligned}
\tag{1.15.2}
$$

$$
\begin{aligned}
Q_{\alpha ab} &= \nabla_{e_\alpha}[g(e_a,e_b)] - g(\nabla_{e_\alpha} e_a, e_b) - g(e_a, \nabla_{e_\alpha} e_b) \\
&= \partial_\alpha(g_{\mu\nu})e_a{}^\mu e_b{}^\nu - e_a{}^\mu\partial_\alpha e^c{}_\mu\eta_{cb} - \omega^c{}_{a\alpha}\eta_{cb} - e_b{}^\mu\partial_\alpha e^c{}_\mu\eta_{ac} - \omega^c{}_{b\alpha}\eta_{ac} \\
&= (\partial_\alpha e^c{}_\mu)e^d{}_\nu\eta_{cd}e_a{}^\mu e_b{}^\nu + e^c{}_\mu(\partial_\alpha e^d{}_\nu)\eta_{cd}e_a{}^\mu e_b{}^\nu - e_a{}^\mu\partial_\alpha e^c{}_\mu\eta_{cb} - \omega^c{}_{a\alpha}\eta_{cb} - e_b{}^\mu\partial_\alpha e^c{}_\mu\eta_{ac} - \omega^c{}_{b\alpha}\eta_{ac} \\
&= \cancel{e_a{}^\mu(\partial_\alpha e^c{}_\mu)\eta_{cb}} + \cancel{e_b{}^\mu(\partial_\alpha e^c{}_\mu)\eta_{ac}} - \cancel{e_a{}^\mu\partial_\alpha e^c{}_\mu\eta_{cb}} - \omega^c{}_{a\alpha}\eta_{cb} - \cancel{e_b{}^\mu\partial_\alpha e^c{}_\mu\eta_{ac}} - \omega^c{}_{b\alpha}\eta_{ac} \\
&= -\omega^c{}_{a\alpha}\eta_{cb} - \omega^c{}_{b\alpha}\eta_{ac},
\end{aligned}
\tag{1.15.3}
$$

where in the second line of eq.(1.15.3) we used $\partial_\alpha(e^c{}_\mu e^d{}_\nu\eta_{cd}) = \partial_\alpha(e^c{}_\mu)e^d{}_\nu\eta_{cd} + e^c{}_\mu\partial_\alpha(e^d{}_\nu)\eta_{cd}$, while the relation $\nabla_{e_\mu}(fX) = e_\mu[f]X + f\nabla_{e_\mu}X$, with $f$ a scalar, has been used multiple times.

## 1.16 Physical interpretation of the spin connection

So far, we have treated the spin connection as a mere tool useful to correct latin indeces in covariant derivatives of mixed tensors, and analyzed its mathematical properties.
Now, we want to address a physical meaning to it. For this purpose, we investigate the behaviour of the spin connection under a local Lorentz tranformation, as previously done for a tetrad basis (1.12.1). Let us first consider an inertial frame $e'^a{}_\mu$ written in a holonomic basis, namely $e'^a{}_\mu = \partial_\mu x'^a$, where $x'^a = x'^a(x^\mu)$ is a Lorentz vector which depends on the spacetime point.
Considering a local Lorentz transformation

$$x'^a \longmapsto x^b = \Lambda^a{}_b(x)x'^a, \tag{1.16.1}$$

we have that

$$e'^{a}{}_{\mu} \longmapsto e^{b}{}_{\mu} = \Lambda^{a}{}_{b}(x) e'^{a}{}_{\mu}. \tag{1.16.2}$$

Let us compute the partial derivative of $x'^a$:

$$\partial_\mu x'^a = \partial_\mu(\Lambda^a{}_b(x)x^b) = (\partial_\mu x^b)\Lambda^a{}_b(x) + x^b(\partial_\mu \Lambda^a{}_b(x)). \tag{1.16.3}$$

On the other hand, it can be written in a slightly different way via a chain rule manipulation:

$$\partial_\mu x'^a = \frac{\partial}{\partial x^\mu} x'^a = \frac{\partial x'^c}{\partial x^\mu}\frac{\partial}{\partial x'^c} x'^a = e'^c{}_\mu \partial'_c x'^a = e'^a{}_\mu = e^c{}_\mu \Lambda^a{}_c(x). \tag{1.16.4}$$

Solving eq.(1.16.4) for $e^c{}_\mu$ and comparing it with eq.(1.16.3) follows that:

$$e^a{}_\mu = (\partial_\mu x^b)\underbrace{\Lambda^a{}_c(x)\Lambda^c{}_b(x)}_{\delta_b{}^a} + \Lambda^a{}_c(x)\partial_\mu \Lambda^c{}_b(x) x^b = \partial_\mu x^a + \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b = \overset{\bullet}{\mathcal{D}}_\mu x^a, \tag{1.16.5}$$

where

$$\boxed{\overset{\bullet}{\omega}{}^a{}_{b\mu} = \Lambda^a{}_c(x)\partial_\mu \Lambda^c{}_b(x)} \tag{1.16.6}$$

is called *purely inertial spin connection* and it represents the inertial effects in a given frame.

However, under local Lorentz transformations the spin connection gauge transforms as [16]:

$$\omega^a{}_{b\mu} = \Lambda^a{}_c(x)\omega'^c{}_{d\mu}\Lambda_b{}^d(x) + \Lambda^a{}_c(x)\partial_\mu \Lambda_b{}^c(x). \tag{1.16.7}$$

Unlike what happens in eq.(1.16.6), in eq.(1.16.7) there are two terms: the first term accounts for non-inertial effects, while the second term captures the inertial effects arising from the rotation of the new frame with respect to the previous one, which occurs when a local Lorentz transformation on a spacetime point is performed [17].

Hence, we understand that the inertial connection (1.16.6) is the outcome of a local Lorentz transformation (1.16.7) when considering a vanishing spin connection $\overset{\bullet}{\omega}{}'^c{}_{d\mu} = 0$.

Thus, starting from an inertial frame where the inertial spin connection vanishes, a local Lorentz transformation $\Lambda^a{}_b(x)$ can always be used to generate different classes of frames. Within each class, the infinitely many frames are related to one another by global Lorentz transformations $\Lambda^a{}_b = \text{const}$.

Coefficients of anholonomy (1.9.10) can be written in terms of spin connection. Let us start by inverting the first line of eq.(1.14.5), namely

$$\begin{aligned} \partial_\mu e^a{}_\sigma &= -e^b{}_\sigma \omega^a{}_{b\mu} + e^b{}_\sigma e^a{}_\lambda e_b{}^\nu \Gamma^\lambda{}_{\mu\nu} \\ &= -e^b{}_\sigma \omega^a{}_{b\mu} + \delta_\sigma{}^\nu e^a{}_\lambda \Gamma^\lambda{}_{\mu\nu}, \end{aligned} \tag{1.16.8}$$

and insert it into the definition of $f^c{}_{ab}$ (1.9.10):

$$\begin{aligned} f^c{}_{ab} &= e_a{}^\mu e_b{}^\nu (\partial_\nu e^c{}_\mu - \partial_\mu e^c{}_\nu) \\ &= e_a{}^\mu e_b{}^\nu (-e^d{}_\nu \omega^c{}_{d\mu} + \cancel{\delta_\mu{}^\alpha e^c{}_\lambda \Gamma^\lambda{}_{\nu\alpha}} + e^d{}_\mu \omega^c{}_{d\nu} - \cancel{\delta_\nu{}^\alpha e^c{}_\lambda \Gamma^\lambda{}_{\mu\alpha}}) \\ &= e_a{}^\mu e_b{}^\nu (-e^d{}_\nu \omega^c{}_{d\mu} + e^d{}_\mu \omega^c{}_{d\nu}) \\ &= -\underbrace{e_b{}^\nu e^d{}_\nu}_{\delta_b{}^d} \omega^c{}_{d\mu} e_a{}^\mu + \underbrace{e_a{}^\mu e^d{}_\mu}_{\delta_a{}^d} \omega^c{}_{d\nu} e_b{}^\nu \\ &= \omega^c{}_{b\mu} e_a{}^\mu - \omega^c{}_{a\nu} e_b{}^\nu \\ &= \omega^c{}_{ba} - \omega^c{}_{ab}. \qquad \text{(using the contraction } \omega^a{}_{bc} = \omega^a{}_{b\mu} e_c{}^\mu) \end{aligned} \tag{1.16.9}$$

Following what we have said before, if we start from an inertial frame, it follows that

$$f^c{}_{ab} = \overset{\bullet}{\omega}{}^c{}_{ba} - \overset{\bullet}{\omega}{}^c{}_{ab}. \tag{1.16.10}$$

From this relation it is possible to define the spin connetion in terms of the coefficients of anholonomy. To do that, it is sufficient to write the coefficients of anholonomy in all the possible combinations, namely, $f^a{}_{bc}$, $f_b{}^a{}_c$ and $f_c{}^a{}_b$. Then, by summing them as $f_b{}^a{}_c + f_c{}^a{}_b - f^a{}_{bc}$, all the terms cancel themeselves out execpt for two identical terms. Thus, we remain with:

$$f_b{}^a{}_c + f_c{}^a{}_b - f^a{}_{bc} = 2\overset{\bullet}{\omega}{}^a{}_{bc}, \tag{1.16.11}$$

from which follows that

$$\boxed{\overset{\bullet}{\omega}{}^a{}_{bc} = \frac{1}{2}\big(f_b{}^a{}_c + f_c{}^a{}_b - f^a{}_{bc}\big)} \tag{1.16.12}$$

In this framework, the curvature tensor accordingly to (1.15.2)

$$R^a{}_{b\mu\nu} = \partial_\nu \overset{\bullet}{\omega}{}^a{}_{b\mu} - \partial_\mu \overset{\bullet}{\omega}{}^a{}_{b\nu} + \overset{\bullet}{\omega}{}^a{}_{d\nu} \overset{\bullet}{\omega}{}^d{}_{b\mu} - \overset{\bullet}{\omega}{}^a{}_{d\mu} \overset{\bullet}{\omega}{}^d{}_{b\nu} = 0. \tag{1.16.13}$$

It is zero due to the property $\Lambda^d{}_c \partial_\mu \Lambda^c{}_d = -\Lambda^c{}_d \partial_\mu \Lambda^d{}_c$. With the same manner, following (1.15.1), the torsion tensor is

$$T^a{}_{\nu\mu} = \partial_\nu e^a{}_\mu - \partial_\mu e^a{}_\nu + \overset{\bullet}{\omega}{}^a{}_{d\nu} e^d{}_\mu - \overset{\bullet}{\omega}{}^a{}_{d\mu} e^d{}_\nu. \tag{1.16.14}$$

This result, physically tells that inertial effects cannot generate "curvature effects", but it is possible to produce only non-null torsional effects. However, if we consider trivial tetrads, namely $e^A{}_\mu = \partial_\mu x^a$ and $\overset{\bullet}{\omega}{}^a{}_{b\mu} = 0$, we can further nullify also the torsion tensor.

# Chapter 2

# General Relativity

General Relativity (GR) is a theory that describes the nature of space, time, and gravity. At its core, the theory presents a simple yet profound idea: gravity is a manifestation of the geometry of spacetime. It is founded on the principle that space and time are entangled into a unified entity known as spacetime, which, in the absence of gravity, simplifies to the flat Minkowski spacetime. This theory is essentially based on the following pillar ideas, which can be stated as follows [17][12][18][19]:

- *The Principle of Relativity* is the requirement that all observers be equally valid for describing physics. In particular, inertial frames (which do not exist globally) are not a priori preferred.
- *The General Covariance Principle* states that the basic laws of Physics can be formulated in tensor form in any smooth four-dimensional manifold $M$. This means that field equations must be "covariant" in form, i.e. they must be invariant under the action of spacetime diffeomorphisms.
- *The Equivalence Principle* (EP) requires acceleration effects to be locally indistinguishable from gravitational effects. More geometrically, in any smooth four-dimensional manifold $M$, it is possible to consider a small spacetime region where spatial and temporal gravitational changes are negligible. Therefore, there always exists a local inertial frame (LIF) where gravitational effects can be nullified.

  The EP states that the *Weak Equivalence Principle* (WEP) — which holds that the motion of an uncharged test body in free fall is independent of its internal structure and composition — is universally valid. Furthermore, the EP ensures that the outcome of any local, non-gravitational experiment is independent of both the velocity of the freely falling apparatus and its location in spacetime.

  There is also an expansion of the EP, called *Strong Equivalence Principle* (SEP), which exihbits the same constraints as the EP, but allows the freely falling bodies to be massive gravitating objects as well as test particles. At the moment, GR is the unique known theory that satisfies SEP.

- *The Causality Principle* requires that each point of spacetime has to admit a universally valid notion of past, present and future.

According to the EP, gravity must be understood as a manifestation of spacetime curvature. This leads to the framework of metric theories of gravity, which are based on the following key postulates:

- Spacetime is equipped with a metric tensor $g_{\mu\nu}$;
- the trajectories of test particles follow geodesics determined by this metric;
- in LIFs, the laws governing non-gravitational physics reduce to those of Special Relativity.

## 2.1 Geodesics equation

### 2.1.1 Geodesics equation from Equivalence Principle

As a consequence of the Equivalence Principle, the free fall motion of a test particle is given by the geodesics equation. In a locally inertial frame (LIF), where the gravitational force is eliminated thanks to the EP, a test particle will draw a straight line, whose equation of motion is given by

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} = 0, \tag{2.1.1}$$

where $ds^2 = \eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$, is the line element. Thus, we are saying that there exists a local coordinate system $y^\mu$ that changes the metric $g_{\mu\nu}$ in a flat Minkowski metric $\eta_{\alpha\beta}$, i.e.[1]

$$\eta_{\alpha\beta} = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta} g_{\mu\nu}. \tag{2.1.2}$$

[1]Geometrically, there always exist two matrices $D^T = \frac{\partial x^i}{\partial y^m}$ and $D = \frac{\partial x^j}{\partial y^n}$ such that $\eta = D^T g D$.

Then, in order to experience gravitational effects, we perform the inverse procedure, which allows us to retrieve the geodesic equation from the Equivalence Principle:

$$\begin{aligned}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right) &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial x^\mu}{\partial y^\nu}\frac{\mathrm{d}y^\nu}{\mathrm{d}s}\right)\\ &= \frac{\partial^2 x^\mu}{\partial s \partial y^\nu}\frac{\mathrm{d}y^\nu}{\mathrm{d}s} + \frac{\partial x^\mu}{\partial y^\nu}\frac{\mathrm{d}^2y^\nu}{\mathrm{d}s^2}\\ &= \frac{\partial^2 x^\mu}{\partial y^\sigma \partial y^\nu}\frac{\mathrm{d}y^\sigma}{\mathrm{d}s}\frac{\mathrm{d}y^\nu}{\mathrm{d}s} + \frac{\partial x^\mu}{\partial y^\nu}\frac{\mathrm{d}^2y^\nu}{\mathrm{d}s^2} = 0. \end{aligned} \tag{2.1.3}$$

Multiply eq.(2.1.3) by $\frac{\partial y^\rho}{\partial x^\mu}$:

$$\underbrace{\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial y^\sigma \partial y^\nu}}_{\mathring{\Gamma}^\rho{}_{\sigma\nu}}\frac{\mathrm{d}y^\sigma}{\mathrm{d}s}\frac{\mathrm{d}y^\nu}{\mathrm{d}s} + \underbrace{\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial x^\mu}{\partial y^\nu}}_{\delta^\rho{}_\nu}\frac{\mathrm{d}^2y^\nu}{\mathrm{d}s^2} = 0.$$

$$\Longrightarrow \boxed{\frac{\mathrm{d}^2y^\rho}{\mathrm{d}s^2} + \mathring{\Gamma}^\rho{}_{\sigma\nu}\frac{\partial y^\sigma}{\partial s}\frac{\partial y^\nu}{\partial s} = 0} \tag{2.1.4}$$

Eq.(2.1.4) is the *geodesic equation* and $\mathring{\Gamma}^\rho{}_{\sigma\nu}$ is called *affine connection* or Christoffel symbols. The affine connection is responsible of the geodesic spacetime structure, which arises from the gravitational force acting on the test particle and being responsible of the departure from the straight trend. From its form,

$$\mathring{\Gamma}^\rho{}_{\sigma\nu} = \frac{\partial y^\rho}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial y^\sigma \partial y^\nu}, \tag{2.1.5}$$

is clear that does not transform as a tensor. Physically they are the apparent forces acting on the body due to the curved geometric background induced by gravity.
From Th.(1.6.2), in a metric compatible and torsion-free spacetime, there exists only one symmetric affine connection, and it is the Levi-Civita. Imposing the condition $\mathring{\nabla}_\rho g_{\sigma\nu} = 0$, we obtain that the Christoffel symbols are the ones defined in (1.6.5), that is

$$\mathring{\Gamma}^\rho{}_{\sigma\nu} := \left\{ \begin{matrix} \rho \\ \sigma\nu \end{matrix} \right\} = \frac{1}{2}g^{\rho\lambda}\big(\partial_\sigma g_{\lambda\nu} + \partial_\nu g_{\sigma\lambda} - \partial_\lambda g_{\sigma\nu}\big) \tag{2.1.6}$$

Thus, the geometry of spacetime is entirely determined by the metric. The metric not only defines distances, but also determines parallel transport through the Levi-Civita connection, which is a connection completely specified by the ten components of the metric tensor (and its derivatives).

**Remark 2.1.1.** For reasons of nomenclature and convention, which will become clear in the following chapters, we will already begin to indicate with *over-circles* the quantities build up on the Levi-Civita connection, e.g. $\mathring{A}^{\mu}{}_{\nu}$.

### 2.1.2 Geodesics equation via the action principle

The geodesic equation can also be retrieved as the equation that minimizes the functional length, which means that it is its extramum via variational principle. Let us start with a curve $\gamma : [\lambda_0, \lambda_1] \subseteq \mathbb{R} \longrightarrow M$ s.t. $\gamma(\lambda_0) = a,\ \gamma(\lambda_1) = b$, then the length of the curve is

$$l = \int_a^b \mathrm{d}s = \int_a^b \sqrt{\mathrm{d}s^2} = \int_a^b \sqrt{-g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu} = \int_{\lambda_0}^{\lambda_1} \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda. \tag{2.1.7}$$

Note that there is a negative sign because we are considering a timelike curve. Its extremum is given by the variational principle:

$$\begin{aligned} 0 = \delta l &= \int_{\lambda_0}^{\lambda_1} \delta\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda = \int_{\lambda_0}^{\lambda_1} \frac{\delta\big(-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\big)}{2\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}}\mathrm{d}\lambda \\ &= \int_{\lambda_0}^{\lambda_1} \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\alpha g_{\mu\nu}\delta x^\alpha + 2g_{\mu\nu}\frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right)\mathrm{d}\tau \end{aligned} \tag{2.1.8}$$

where

$$\mathrm{d}\tau = \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda. \tag{2.1.9}$$

We now integrate by parts, remembering that the total derivative is zero at the boundaries:

$$\begin{aligned} 0 &= \int_{\lambda_0}^{\lambda_1} \left[\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\alpha g_{\mu\nu}\delta x^\alpha - 2\delta x^\mu \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right)\right]\mathrm{d}\tau \\ &= \int_{\lambda_0}^{\lambda_1} \left[\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\alpha g_{\mu\nu}\delta x^\alpha - 2\delta x^\mu \partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} - 2\delta x^\mu g_{\mu\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}\right]\mathrm{d}\tau \\ &= \int_{\lambda_0}^{\lambda_1} \left[-2g_{\mu\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2} + \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\mu g_{\alpha\nu} - \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\alpha g_{\mu\nu} - \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\partial_\nu g_{\mu\alpha}\right]\delta x^\mu \mathrm{d}\tau, \\ &= \int_{\lambda_0}^{\lambda_1} \left[g_{\mu\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2} + \frac{1}{2}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\big(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}\big)\right]\delta x^\mu \mathrm{d}\tau. \end{aligned} \tag{2.1.10}$$

Taking an arbitrary variation $\delta x^\mu$ and multiplying by the inverse metric tensor $g^{\mu\beta}$ we obtain

$$\frac{\mathrm{d}^2 x^\beta}{\mathrm{d}\tau^2} + \mathring{\Gamma}^\beta{}_{\alpha\nu}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0, \tag{2.1.11}$$

where

$$\mathring{\Gamma}^\beta{}_{\alpha\nu} = \frac{1}{2}g^{\mu\beta}\big(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}\big) \tag{2.1.12}$$

are the Christoffel symbols.

It is worth noticing that in GR, the autoparallel equation and geodesic equation coincide.

### 2.1.3 The Riemann curvature tensor

We have observed how geometric curvature influences the geodesic equation, but to describe it quantitatively as a field, we need to introduce the *Riemann curvature tensor* $\mathring{R}^\mu{}_{\nu\rho\sigma}$.
We saw that the condition on the Levi-Civita connection, in terms of tensors (1.7.1), reads as

$$\mathring{T}^\rho{}_{\mu\nu} = \mathring{\Gamma}^\rho{}_{\mu\nu} - \mathring{\Gamma}^\rho{}_{\nu\mu} \implies \mathring{T}^\rho{}_{\mu\nu} = 0. \tag{2.1.13}$$

Then, since both non-metricity tensor and torsion tensor are zero in GR, the gravitational field is only described by curvature, whose expression is given the commutation of covariant derivatives on a generic vector $v^\mu$, namely

$$[\mathring{\nabla}_\rho, \mathring{\nabla}_\sigma]v^\mu = \mathring{R}^\mu{}_{\nu\rho\sigma}v^\nu, \tag{2.1.14}$$

whose extended form reads as (cfr eq.(1.7.3)):

$$\mathring{R}^\mu{}_{\nu\rho\sigma} = \partial_\rho\mathring{\Gamma}^\mu{}_{\nu\sigma} - \partial_\sigma\mathring{\Gamma}^\mu{}_{\nu\rho} + \mathring{\Gamma}^\mu{}_{\alpha\rho}\mathring{\Gamma}^\alpha{}_{\nu\sigma} - \mathring{\Gamma}^\mu{}_{\alpha\sigma}\mathring{\Gamma}^\alpha{}_{\nu\rho}. \tag{2.1.15}$$

The above equation indicates that Schwarz's theorem does not hold when applied to the covariant derivative—unless the spacetime is flat (i.e. $\mathring{R}^\mu{}_{\nu\rho\sigma} = 0$).
The Riemann tensor mantains its general property (1.7.8), which in this case corresponds to $\mathring{R}^\mu{}_{\nu\rho\sigma} = -\mathring{R}^\mu{}_{\nu\sigma\rho}$. Moreover, in the case of GR, due to the symmetries of the Levi-Civita

connection, it acquires the following further symmetries:

$$\mathring{R}_{\mu\nu\rho\sigma} = -\mathring{R}_{\nu\mu\rho\sigma}; \tag{2.1.16}$$

$$\mathring{R}_{\mu\nu\rho\sigma} = \mathring{R}_{\rho\sigma\mu\nu}. \tag{2.1.17}$$

In the GR framework, the Bianchi's identities have both the right members equal to zero, since it is torsion-free. Hence eqs.(1.7.10) and (1.7.11) become:

$$\mathring{R}^{\mu}{}_{[\nu\rho\sigma]} = \mathring{R}^{\mu}{}_{\nu\rho\sigma} + \mathring{R}^{\mu}{}_{\rho\sigma\nu} + \mathring{R}^{\mu}{}_{\sigma\nu\rho} = 0; \tag{2.1.18}$$

$$\mathring{\nabla}_{[\alpha}\mathring{R}^{\mu}{}_{|\nu|\rho\sigma]} = \mathring{\nabla}_{\alpha}\mathring{R}^{\mu}{}_{\nu\rho\sigma} + \mathring{\nabla}_{\nu}\mathring{R}^{\mu}{}_{\rho\alpha\sigma} + \mathring{\nabla}_{\rho}\mathring{R}^{\mu}{}_{\alpha\nu\sigma} = 0. \tag{2.1.19}$$

Due to the symmetries (2.1.16), we can define the

- *symmetric Ricci tensor*

$$\mathring{R}_{\mu\nu} = \mathring{R}^{\alpha}{}_{\mu\alpha\nu}; \tag{2.1.20}$$

- *scalar curvature*

$$\mathring{R} = \mathring{R}^{\mu}{}_{\mu}. \tag{2.1.21}$$

## 2.2 GR field equations

GR field equations can be obtained in different ways. Before deriving Eintein's equations via the GR action, we want to present a heuristic[2] approach to retrieve them.

Let us start by wondering what equation describes the relation between spacetime geometry and the matter distribution. An important clue is provided by the comparison of the description of tidal force in Newtonian gravity and GR. In the Newtonian theory, the gravitational field may be represented by a potential $\phi$, and the tidal acceleration of two nearby particles is given by

$$\bar{A} = -(\bar{x} \cdot \bar{\nabla})\bar{\nabla}\phi, \tag{2.2.1}$$

[2] Einstein's equations cannot be derived from other equations.

where $\bar{x}$ is the separation vector between the two particles. On the other hand, in GR, from eq.(1.8.9), the tidal acceleration of two nearby particles is given by

$$A^\mu = \mathring{R}^\mu{}_{\sigma\lambda\nu} V^\sigma V^\lambda X^\nu, \tag{2.2.2}$$

where $V^\mu$ is the four-velocity of the particles and $X^\nu$ is the deviation vector. This suggests we can make the following correspondence:

$$\mathring{R}^\mu{}_{\sigma\lambda\nu} V^\sigma V^\lambda \longleftrightarrow -\partial_\nu \partial^\mu \phi. \tag{2.2.3}$$

However, we want to find an equation that substitutes the Poisson equation of a Newtonian potential, namely

$$\nabla^2 \phi = \frac{4\pi G}{c^4} \rho, \tag{2.2.4}$$

where $\rho$ is the energy-mass density of matter. Furthermore, we know that in GR the energy properties of matter are described by a stress-energy tensor $T_{\mu\nu}$ that satisfies $\mathring{\nabla}_\mu T^{\mu\nu} = 0$. Hence, an observer with four-velocity $V$ measures the energy density as[3]

$$\rho = T_{\sigma\lambda} V^\sigma V^\lambda. \tag{2.2.5}$$

Observe now that $\nabla^2\phi = \partial_\mu \partial^\mu \phi$, hence from eq.(2.2.3), we obtain

$$\mathring{R}^\mu{}_{\sigma\lambda\mu} V^\sigma V^\lambda = -4\pi T_{\sigma\lambda} V^\sigma V^\lambda, \tag{2.2.6}$$

which can be written as

$$\mathring{R}_{\sigma\lambda} = 4\pi T_{\sigma\lambda}. \tag{2.2.7}$$

Hence, we have arrived at an equation in which the stress-energy tensor is proportional to a symmetric tensor constructed from the second derivatives of the metric.

Eq.(2.2.7) are the equations firstly postulated by Einstein. However, they present a problem: from the second Bianchi identity we have that the divergence of the Einstein

[3]The stress-energy tensor is $T_{\mu\nu} = Pg_{\mu\nu} + (p+P)V_\mu V_\nu$, but due to the Newtonian limit the pressure is neglected since $v \ll c$.

tensor vanishes, namely

$$\mathring{\nabla}_\mu \mathring{G}^{\mu\nu} = \mathring{\nabla}_\mu \big(\mathring{R}^{\mu\nu} - \frac{1}{2}\mathring{R} g_{\mu\nu}\big) = 0$$
$$\overset{(2.2.7)}{\Longrightarrow} \underbrace{\mathring{\nabla}_\mu R^{\mu\nu}}_{4\pi\mathring{\nabla}_\mu T^{\mu\nu}=0} - \frac{1}{2} g^{\mu\nu}\mathring{\nabla}_\mu \mathring{R} = 0 \implies \partial_\mu \mathring{R} = 0 \tag{2.2.8}$$

due to the energy conservation of the stress-energy tensor. Thus, this calculation wuold imply that the scalar curvature $\mathring{R}$ is constant in the universe, as well as $T = T^\mu{}_\mu$. Clearly, these implication about the mass distribution are not physical, hence it is necessary to find another tensor to replace $R_{\sigma\mu}$: the other symmetric 2-tensor constructed from the Ricci tensor is $\mathring{G}_{\mu\nu}$. Taking into account the divergenceless behaviour of $\mathring{G}_{\mu\nu}$, let us consider the following equations

$$\boxed{\mathring{G}_{\mu\nu} = \mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}. \tag{2.2.9}$$

Eq.(2.2.9) are the *Einstein's equations* and do not present inconsistency between the Bianchi identity and the energy conservation, in fact

$$\mathring{\nabla}_\mu \mathring{G}^{\mu\nu} = 0 \implies \frac{8\pi G}{c^4} \mathring{\nabla}_\mu T^{\mu\nu} = 0. \tag{2.2.10}$$

Thus, the energy conservation is now a consequence of the second Bianchi identity.
It is also possible to prove that the correspondeces which have been used to obtain eq.(2.2.7) are valid for eq.(2.2.9) too.

The entire content of General Relativity can be summarized as follows:

- the spacetime is a differentiable manifold equipped with a Lorentzian metric $g_{\mu\nu}$;
- the curvature of $g_{\mu\nu}$ is related to the matter distribution in the Einstein's equations (2.2.9).

### 2.2.1 Derivation from the Einstein-Hilbert action

We now derive the Einstein's field equations starting from the Einstein-Hilber action, i.e.

$$S_{EH} = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g}\mathring{R}. \tag{2.2.11}$$

Note that $\mathring{R}$ is the only scalar that can be constructed from the Riemann tensor which leads to second-order equations in the metric[4].
Using $\mathring{R} = g^{\mu\nu}\mathring{R}_{\mu\nu}$, the variation of (2.2.11) is

$$\delta S_{EH} = \frac{c^4}{16\pi G}\int d^4x\left[\sqrt{-g}g^{\mu\nu}\delta\mathring{R}_{\mu\nu} + \sqrt{-g}\mathring{R}_{\mu\nu}\delta g_{\mu\nu} + \mathring{R}\delta\sqrt{-g}\right]. \tag{2.2.12}$$

The first term of (2.2.12) could be simplified. To do so, let us recall that under a metric varitation, the connection $\mathring{\Gamma}^{\rho}{}_{\mu\nu}$ varies as

$$\mathring{\Gamma}^{\rho}{}_{\mu\nu} \longrightarrow \mathring{\Gamma}^{\rho}{}_{\mu\nu} + \delta\mathring{\Gamma}^{\rho}{}_{\mu\nu}. \tag{2.2.13}$$

Note that even if the connection does not transform as a tensor, a difference between two connections do. Thus, we can compute its covariant derivative:

$$\mathring{\nabla}_{\lambda}\delta\mathring{\Gamma}^{\rho}{}_{\nu\mu} = \partial_{\lambda}\delta\mathring{\Gamma}^{\rho}{}_{\nu\mu} + \mathring{\Gamma}^{\rho}{}_{\lambda\sigma}\delta\mathring{\Gamma}^{\sigma}{}_{\nu\mu} - \mathring{\Gamma}^{\sigma}{}_{\lambda\nu}\delta\mathring{\Gamma}^{\rho}{}_{\sigma\mu} - \mathring{\Gamma}^{\sigma}{}_{\lambda\mu}\delta\mathring{\Gamma}^{\rho}{}_{\nu\sigma}. \tag{2.2.14}$$

Remembering eq.(1.7.4), we can write the variation of the Riemann tensor using (2.2.14) as

$$\delta\mathring{R}^{\rho}{}_{\mu\lambda\nu} = \mathring{\nabla}_{\lambda}\delta\mathring{\Gamma}^{\rho}{}_{\nu\mu} - \mathring{\nabla}_{\nu}\delta\Gamma^{\rho}{}_{\lambda\mu}. \tag{2.2.15}$$

Then, the first term of (2.2.12) becomes

$$\begin{aligned}
\frac{c^4}{16\pi G}\int d^4x\sqrt{-g}g^{\mu\nu}\delta\mathring{R}_{\mu\nu} &= \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}g^{\mu\nu}\left[\mathring{\nabla}_{\lambda}\delta\mathring{\Gamma}^{\lambda}{}_{\nu\mu} - \mathring{\nabla}_{\nu}\delta\Gamma^{\lambda}{}_{\lambda\mu}\right] \\
&= \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\left[\mathring{\nabla}_{\lambda}\big(g^{\mu\nu}\delta\mathring{\Gamma}^{\lambda}{}_{\nu\mu}\big) - \mathring{\nabla}_{\lambda}\big(g^{\mu\lambda}\delta\Gamma^{\sigma}{}_{\sigma\mu}\big)\right] \\
&= \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\mathring{\nabla}_{\lambda}v^{\lambda} \\
&= \frac{c^4}{16\pi G}\int d^4x\ \partial_{\lambda}(\sqrt{-g}\ v^{\lambda}).
\end{aligned} \tag{2.2.16}$$

Hence, this term is a total derivative, and invoking the Stoke's theorem, it gives a boundary term when integrated, which does not contribute to the field equations.

Now, we will manipulate the third term of (2.2.12). We use the formula

$$\mathrm{tr}(\ln\mathrm{M}) = \ln(\det\mathrm{M}), \tag{2.2.17}$$

[4]We may write $\int\sqrt{-g}\mathring{R}_{\mu\nu}\mathring{R}^{\mu\nu}$, but the Ricci tensor is of the second order, hence after integration by parts we will obtain fourth-order equations, which contain Ghost fields and tachyonic particles.

which holds for every invartible matrix M. Its variation yields

$$\mathrm{trM}^{-1}\delta\mathrm{M} = \frac{1}{\mathrm{detM}}\delta(\mathrm{detM}). \tag{2.2.18}$$

Setting $\mathrm{M} = g^{\mu\nu}$ we have that eq.(2.2.18) beacomes

$$g_{\mu\nu}\delta g^{\mu\nu} = g\delta g^{-1}. \tag{2.2.19}$$

Hence,

$$\begin{aligned}\delta\sqrt{-g} = \delta[((-g)^{-1})^{-1/2}] &= -\frac{1}{2}(-g^{-1})^{-3/2}\delta(-g^{-1}) \\ &= -\frac{1}{2}(-g^{-1})^{-3/2}(-g^{-1})g_{\mu\nu}\delta g^{\mu\nu} \\ &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}.\end{aligned} \tag{2.2.20}$$

We can finally rewrite the variation of the action (2.2.12) as

$$\delta S_{EH} = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\left[\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu}\right]\delta g_{\mu\nu}. \tag{2.2.21}$$

Imposing $\delta S_{EH} = 0$, we get

$$\mathring{G}_{\mu\nu} = \mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu} = 0, \tag{2.2.22}$$

which are the vacuum Einstein's equations.

In order to obtain the Einstein's equations in presence of matter, it is sufficient to add the mass action, namely

$$S_{GR} = S_{EH} + S_m, \tag{2.2.23}$$

which yelds

$$\delta S_{GR} = \delta S_{EH} + \delta S_m = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\mathring{G}_{\mu\nu}\delta g^{\mu\nu} + \int d^4x\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}}\delta g^{\mu\nu}, \tag{2.2.24}$$

where we can define from the last term the energy-momentum tensor of matter[5] as

$$\mathfrak{T}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}}, \tag{2.2.25}$$

which is symmetric, satisfies the conservation equations $\mathring{\nabla}_\mu \mathfrak{T}^{\mu\nu} = 0$ (see later), and physically represents the source of gravitational field. Again, from $\delta S_{GR} = 0$, we get

$$\boxed{\mathring{G}_{\mu\nu} - \frac{8\pi G}{c^4}\mathfrak{T}_{\mu\nu} = 0}. \tag{2.2.26}$$

Let us show explicitly the conservation of the energy-momentum tensor $\mathfrak{T}_{\mu\nu}$. We start from the second Bianchi identity (1.7.11) with $R^\alpha{}_{\beta\mu\nu} = \mathring{R}^\alpha{}_{\beta\mu\nu}$ and $T^\alpha{}_{\beta\gamma} = Q_{\alpha\beta\gamma} = 0$ since we are dealing with GR. Hence we remain with

$$\mathring{\nabla}_\lambda \mathring{R}^\alpha{}_{\beta\mu\nu} + \mathring{\nabla}_\mu \mathring{R}^\alpha{}_{\beta\nu\lambda} + \mathring{\nabla}_\nu \mathring{R}^\alpha{}_{\beta\lambda\mu} = 0. \tag{2.2.27}$$

We can further simplify this equation by invoking the Covariance Principle, which allows us to choose a Local Inertial Frame (LIF), where the first derivatives of the metric vanish, while the second derivatives generally do not. Thus, eq.(2.2.27) becomes

$$\partial_\lambda \mathring{R}^\alpha{}_{\beta\mu\nu} + \partial_\mu \mathring{R}^\alpha{}_{\beta\nu\lambda} + \partial_\nu \mathring{R}^\alpha{}_{\beta\lambda\mu} = 0. \tag{2.2.28}$$

Contract now $\alpha$ with $\lambda$ and exploit the antisymmetry of the Riemann tensor (see (2.1.16)):

$$\partial_\lambda \mathring{R}^\lambda{}_{\beta\mu\nu} + \partial_\mu \mathring{R}^\lambda{}_{\beta\nu\lambda} + \partial_\nu \mathring{R}^\lambda{}_{\beta\lambda\mu} = 0, \tag{2.2.29}$$

and raising the index $\beta$ and contracting it with $\mu$, it follows that

$$-\partial_\lambda \mathring{R}^\lambda{}_\nu - \partial_\beta \mathring{R}^\beta{}_\nu + \partial_\nu \mathring{R} = 0 \implies \partial_\mu \mathring{R}^\mu{}_\nu - \frac{1}{2}\partial_\nu \mathring{R} = 0 \tag{2.2.30}$$

From eq.(2.2.30) we obtain

$$\partial_\mu\big(\mathring{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathring{R}\big) = 0 \implies \mathring{\nabla}_\mu\big(\mathring{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathring{R}\big) = 0. \tag{2.2.31}$$

[5] Note that this definition of energy-momentum tensor could differ from the one constructed from the Noether theorem.

Hence, we have shown the divergenceless of the Einstein tensor $\mathring{G}_{\mu\nu}$, and as consequence we have that

$$\mathring{\nabla}_\mu \mathring{G}^{\mu\nu} = 0 \implies \mathring{\nabla}_\mu \mathfrak{T}^{\mu\nu} = 0. \tag{2.2.32}$$

Let us now talk about the GR degrees of freedom. In GR the gravitational field is described by the metric tensor $g_{\mu\nu}$, which is a symmetric $4 \times 4$ tensor. This symmetry implies that $g_{\mu\nu}$ has 10 independent components, since $(4 \cdot (4+1))/2 = 10$. However, not all of these components correspond to physical DoFs. This is due to the diffeomorphism invariance, which reflects the freedom to perform arbitrary smooth coordinate transformations. These transformations depend on 4 arbitrary functions (one for each coordinate), which removes 4 DoFs associated with gauge redundancy. Additionally, we can impose 4 gauge-fixing conditions by choosing a specific coordinate system, which eliminates another 4 components. Therefore, the number of true physical (dynamical) DoFs is

$$10 \text{ (components of } g_{\mu\nu}) - 4 \text{ (gauge freedom)} - 4 \text{ (gauge fixing)} = 2.$$

These two remaining degrees of freedom correspond to the two polarization modes of gravitational waves: the "plus" (+) and "cross" (×) modes.
In the context of quantum field theory, this result is consistent with the properties of a massless spin-2 particle – the graviton – which also has two polarization states in four-dimensional spacetime.

At this point, it is important to make a fundamental observation: GR cannot be considered a gauge theory.
In theoretical physics, the term *gauge theory* can be understood in two distinct ways. In a broad sense, it refers to the presence of internal symmetries or redundancies in a physical theory – symmetries that do not affect observable quantities. For instance, in electromagnetism, the gauge freedom in the vector potential $A_\mu \to A_\mu + \partial_\mu \chi$ leaves the physical fields unchanged.

However, in the more rigorous and geometrical formulation used in modern particle physics, a gauge theory is described in terms of fiber bundles, where a connection 1-form $\omega$ defines a covariant derivative $D$, and the field strength is given by $\Omega = D\omega$, representing the curvature of the bundle (see Sec.(1.9)). These gauge theories, such as those based on the groups SU(2), SU(3), and U(1), form the backbone of the Standard Model. GR, on the other hand, is not a gauge theory in this strict sense. Although it possesses diffeomorphism invariance (coordinate freedom) and can be reformulated using geometrical tools reminiscent of gauge theory (e.g. in the tetrad formalism), its symmetries are not internal but are instead related to the structure of spacetime itself. Another fundamental difference between the tetrad formalism and gauge theory lies in the nature of their internal spaces. In gauge theories, the internal space is independent of the base manifold, typically representing an abstract internal symmetry group.

In contrast, in Riemannian (or Lorentzian) geometry, the tetrad formalism introduces an internal Minkowski space at each point that is canonically identified with the tangent space of the manifold. Through the tetrad field, this internal space is intrinsically linked to the geometry of spacetime, so that once the manifold and its metric structure are specified, the associated tangent space structures are naturally determined.
Thus, the internal freedoms in GR cannot be expressed through a fiber bundle structure with a gauge connection and curvature in the same way as in particle physics.
The key distinctions from gauge theory are outlined below [16]:

- In gauge theories, the fundamental field – used for defining variations – is a connection or gauge potential. In contrast, in GR, this role is played by the metric tensor.
- While GR does involve a connection (the Levi-Civita connection), it is not treated as a fundamental variable. Once a metric is specified, the connection is uniquely determined. Furthermore, the Levi-Civita connection is not a true gravitational variable in the classical field sense, nor is it a genuine gravitational connection, as it encodes both gravitational and inertial effects.
- Gauge theory Lagrangians are typically quadratic in curvature (e.g., in QED we have the term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$; see later (4.3.44)) . By contrast, the EH Lagrangian of GR is linear in the curvature scalar.
- There is no scalar quantity in GR that can be constructed solely from the metric and its first derivatives. The EH Lagrangian, in fact, depends on the second derivatives of the metric, in addition to the metric and its first derivative.
- In gauge theories, interactions manifest as forces. In GR, however, gravitation appears as a geometric phenomenon – there is no gravitational force in the traditional sense.

We will see in Chap.(4) how we can formulate a gauge theory of gravity.

# Chapter 3

# Metric-Affine theories of Gravity

To introduce this argument, we start with the following question:

*Which connection do we choose on a given manifold?*

In general, given a differential structure on the manifold, there are different covariant derivatives, and none of these stand out in particular.
In fact, it is possible to introduce a connection without defining a metric. However, if a metric $g$ is already given, then there exists a natural choice for the connection $\nabla$: the Levi-Civita connection. As we saw in Chap.(2), this is precisely the case in General Relativity, where the geometry of spacetime is entirely determined by the metric. The metric not only defines distances, but also determines parallel transport through the Levi-Civita connection, which is a connection completely specified by the ten components of the metric tensor (and its derivatives) (cfr eq.(1.6.4)):

$$\mathring{\Gamma}^{\rho}{}_{\mu\nu} := \left\{ \begin{array}{c} \rho \\ \mu\nu \end{array} \right\} = \frac{1}{2} g^{\rho\lambda} \big( \partial_{\mu} g_{\lambda\nu} + \partial_{\nu} g_{\mu\lambda} - \partial_{\lambda} g_{\mu\nu} \big) \tag{3.0.1}$$

Moreover, we saw that Levi-Civita connection is **torsion-less** and **metric compatible**, thus the gravitational field is only described by curvature.

However, we can investigate other possible ways in which the gravity can be geometrized, and a first extension of the General Relativity starts by generalizing the affine connection, which can be different from the Levi-Civita's, giving rise to metric-affine theories.

A **metric-affine theory** is a triplet $\{M, g_{\mu\nu}, \Gamma^{\rho}{}_{\mu\nu}\}$, where $(M, g_{\mu\nu})$ is a Lorentzian manifold (see Def.(1.5.1)), while $\Gamma^{\rho}{}_{\mu\nu}$ is the affine connection, endowed with 64 independent components. $g_{\mu\nu}$ and $\Gamma^{\rho}{}_{\mu\nu}$ are now completely independent[1].
Dealing with a metric-affine theory means that the tensors defined previously in Sec.(1.7),

[1]This holds if we do not consider the Equivalence principle as founding hypothesis. (see Chap.(2))

i.e. the Curvature tensor $R^{\mu}{}_{\nu\rho\sigma}$, the tosion tensor $T^{\mu}{}_{\nu\rho}$ and the non-metricity tensor $Q_{\mu\nu\rho}$, are non-zero a priori. In addition to the curvature of spacetime, the presence of the other two tensors introduces unusual effects on its geometry. These effects can be understood by visualizing how the parallel transport of a vector on a manifold is influenced.

- *Curvature* reveals itself when a vector is parallel transported around a closed loop in a non-flat space, returning to its starting point with a different orientation than it began: (Fig.(3.1A))
- *Torsion* implies a rotational geometry in which the parallel transport of two vectors becomes antisymmetric when the vectors and the direction of transport are exchanged. This leads to the failure of parallelograms to close: (Fig.(3.1B))
- *Non-metricity* is responsible for variations in vector length during parallel transport: (Fig.(3.1C))

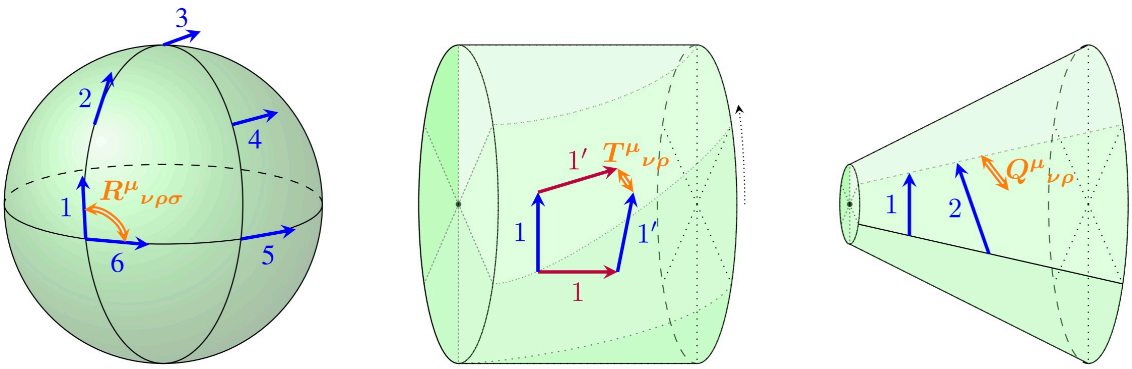



FIGURE 3.1: Geometrical representation of Curvature (A), Torsion (B) and Non-metricity (C), respectively. Figure credits [13].

## 3.1 Generalized affine connection

Due to the introduction of the torsion and non-metricity tensors, the connection on the manifold cannot be the Leci-Civita's (3.0.1) as before. From the fact that $Q_{\rho\mu\nu} = \nabla_{\rho} g_{\mu\nu} \neq 0$, we can build an equation usign $Q_{\rho\mu\nu}$ and the other two combinations, $Q_{\nu\rho\mu}$ and $Q_{\mu\nu\rho}$ [20][21]:

$$Q_{\nu\rho\mu} + Q_{\mu\nu\rho} - Q_{\rho\mu\nu} = \nabla_{\nu} g_{\rho\mu} + \nabla_{\mu} g_{\nu\rho} - \nabla_{\rho} g_{\mu\nu},$$
$$\implies \nabla_{\rho} g_{\mu\nu} - \nabla_{\nu} g_{\rho\mu} - \nabla_{\mu} g_{\nu\rho} + Q_{\nu\rho\mu} + Q_{\mu\nu\rho} - Q_{\rho\mu\nu} = 0. \tag{3.1.1}$$

Substituting the covariant derivative of the metric, i.e.

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda{}_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda{}_{\rho\nu} g_{\mu\lambda}, \tag{3.1.2}$$

in eq.(3.1.1), it follows that

$$\begin{aligned} \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} - \underline{\Gamma^\lambda{}_{\rho\mu} g_{\lambda\nu}} - \Gamma^\lambda{}_{\rho\nu} g_{\mu\lambda} + \Gamma^\lambda{}_{\mu\nu} g_{\lambda\rho} \\ + \underline{\Gamma^\lambda{}_{\mu\rho} g_{\nu\lambda}} + \Gamma^\lambda{}_{\nu\rho} g_{\lambda\mu} + \Gamma^\lambda{}_{\nu\mu} g_{\rho\lambda} - Q_{\rho\mu\nu} + Q_{\nu\rho\mu} + Q_{\mu\nu\rho} = 0. \end{aligned} \tag{3.1.3}$$

Using eq.(1.7.2), i.e. $T^\lambda{}_{\mu\rho} = \Gamma^\lambda{}_{\mu\rho} - \Gamma^\lambda{}_{\rho\mu}$, on the underlined terms and adding and subtracting a $\Gamma^\lambda{}_{\mu\nu} g_{\rho\lambda}$ term, we obtain

$$\begin{aligned} \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + g_{\nu\lambda} T^\lambda{}_{\mu\rho} + g_{\lambda\mu} T^\lambda{}_{\nu\rho} + g_{\lambda\rho} \big( \Gamma^\lambda{}_{\mu\nu} + \Gamma^\lambda{}_{\nu\mu} \big) \\ + \Gamma^\lambda{}_{\mu\nu} g_{\rho\lambda} - \Gamma^\lambda{}_{\mu\nu} g_{\rho\lambda} - Q_{\rho\mu\nu} + Q_{\nu\rho\mu} + Q_{\mu\nu\rho} = 0. \end{aligned} \tag{3.1.4}$$

With the same passages, we obtain finally

$$2 g_{\rho\lambda} \Gamma^\lambda{}_{\mu\nu} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + g_{\nu\lambda} T^\lambda{}_{\mu\rho} + g_{\lambda\mu} T^\lambda{}_{\nu\rho} + g_{\rho\lambda} T^\lambda{}_{\nu\mu} - Q_{\rho\mu\nu} + Q_{\nu\rho\mu} + Q_{\mu\nu\rho} = 0. \tag{3.1.5}$$

Contracting eq.(3.1.5) with $g^{\rho\alpha}$, a general connection acquires the following form;

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} \big( \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} \big) - \frac{1}{2} \big( T_{\nu\mu}{}^\alpha + T_{\mu\nu}{}^\alpha + \underbrace{T^\alpha{}_{\nu\mu}}_{-T^\alpha{}_{\mu\nu}} \big) + \frac{1}{2} \big( Q^\alpha{}_{\mu\nu} - \underbrace{Q_{\mu\nu}{}^\alpha}_{Q_\mu{}^\alpha{}_\nu} - Q_\nu{}^\alpha{}_\mu \big). \tag{3.1.6}$$

Hence, a general connection $\Gamma^\rho_{\mu\nu}$ admits the following decomposition.

$$\boxed{\Gamma^\rho{}_{\mu\nu} := \mathring{\Gamma}^\rho{}_{\mu\nu} + K^\rho{}_{\mu\nu} + L^\rho{}_{\mu\nu}}, \tag{3.1.7}$$

where $K^\rho{}_{\mu\nu}$ is the *Contortion tensor* and $L^\rho{}_{\mu\nu}$ the *Disformation tensor*, defined as

$$K^\rho{}_{\mu\nu} := \frac{1}{2} (T_\mu{}^\rho{}_\nu + T_\nu{}^\rho{}_\mu - T^\rho{}_{\mu\nu}), \tag{3.1.8}$$

$$L^\rho{}_{\mu\nu} := \frac{1}{2} (Q^\rho{}_{\mu\nu} - Q_\mu{}^\rho{}_\nu - Q_\nu{}^\rho{}_\mu). \tag{3.1.9}$$

The contortion tensor is antisymmetric under the interchange of the first and third index, namely

$$K^{\rho}{}_{\mu\nu} = -K_{\nu\mu}{}^{\rho}, \tag{3.1.10}$$

while the disformation tensor is symmetric under the interchange of the last two indeces:

$$L^{\rho}{}_{\mu\nu} = L^{\rho}{}_{\nu\mu}. \tag{3.1.11}$$

From this, we can write the covariant derivative of a generic $(1,1)$-tensor:

$$\nabla_{\mu} A^{\alpha}{}_{\beta} := \partial_{\mu} A^{\alpha}{}_{\beta} - \Gamma^{\rho}{}_{\beta\mu} A^{\alpha}{}_{\rho} + \Gamma^{\alpha}{}_{\rho\mu} A^{\rho}{}_{\beta}. \tag{3.1.12}$$

## 3.2 Hypermomentum, canonical and metrical energy-momentum tensor

In MAG theories, the action $S$ is a functional of the metric $g$, the generalised affine connection $\Gamma$ and a class of matter fields $\phi_a$, namely

$$S[g, \Gamma, \phi] = S_g[g, \Gamma] + S_m[g, \Gamma, \phi], \tag{3.2.1}$$

where

$$S_g[g, \Gamma] = \frac{c^4}{16\pi G} \int d^n x \sqrt{-g} \mathcal{L}_g(g, \Gamma), \tag{3.2.2}$$

$$S_m[g, \Gamma . \phi] = \int d^n x \sqrt{-g} \mathcal{L}_m(g, \Gamma, \phi) \tag{3.2.3}$$

represent the gravitational and the matter sector, repsectively. Making use of the matter action, it is possible to define the *canonical energy-momentum tensor*

$$\Sigma^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}_m}{\partial \nabla_{\mu} \phi^a} \nabla_{\nu} \phi^a - \delta^{\mu}{}_{\nu} \mathcal{L}_m, \tag{3.2.4}$$

and the *(metrical) energy-momentum tensor*

$$\mathfrak{T}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}, \tag{3.2.5}$$

which is symmetric.

From the coupling between the connection and the matter naturally arises the *hypermomentum tensor*[22], defined as

$$H_\alpha{}^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta \Gamma^\alpha{}_{\mu\nu}} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta \Gamma^\alpha{}_{\mu\nu}}. \tag{3.2.6}$$

In the context of MAG, in which we have a non-Riemann geometry, the hypermomentum generalizes the usual energy-momentum tensor. In fact, just as the stress-energy tensor in GR describes the density and flux of energy and momentum and couples to the metric, the hypermomentum represents the density of internal currents associated with the microstructure of spacetime (or of a continuous medium) and couples to the affine connection. Formally, the hypermomentum density can be decomposed as [23]

$$\text{hypermomentum density} = \text{spin current} \oplus \text{dilation current} \oplus \text{shear current}.$$

Physically, this means that:

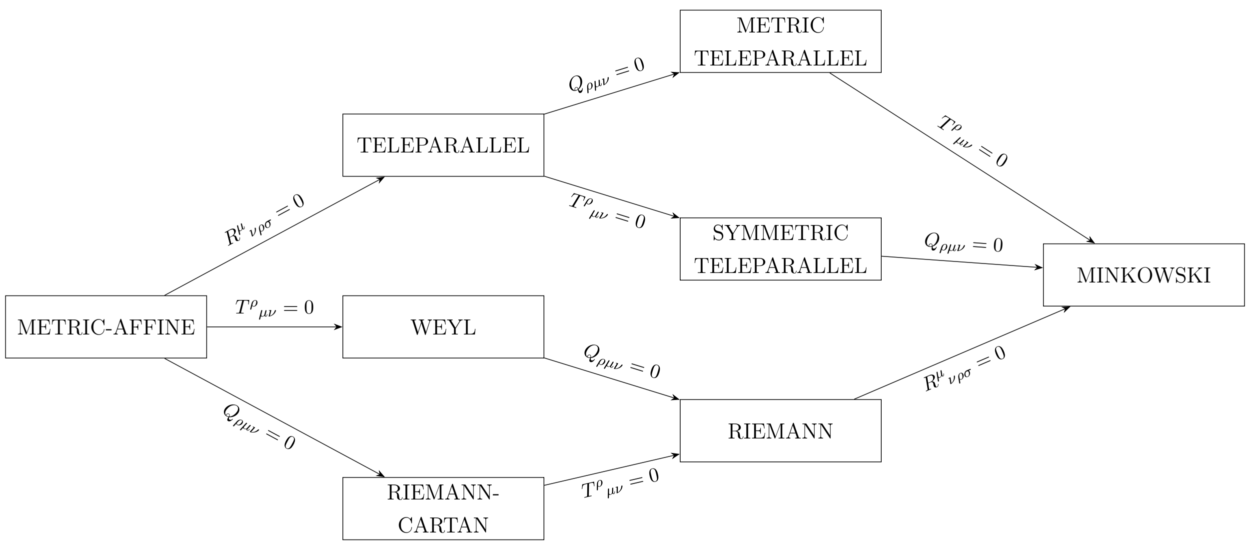
- the *spin current* corresponds to spacetime *torsion*;
- the *dilation current* corresponds to the *volumetric part* of nonmetricity (changes of scale);
- the *shear current* corresponds to the *deviatoric part* of nonmetricity (shape distortions).

## 3.3 Metric-Affine subclasses

So far we have seen that the most general metric-affine theory is defined by a Lorentzian manifold endowed with a general affine connection and the presence of all the tensorial effects given by curvature, torsion and non-metricity. By demanding that certain of these tensorial quantities vanish, we retrieve different subclasses of the general metric-affine structure. We willl list the principal cases below[2]:

[2]We will refer to objects related to general affine connection by denoting them without any overscripts.

- $R^\mu{}_{\nu\rho\sigma} = 0$. In the case of vanishing curvature, the connection is said to be *flat*. If both torsion and non-metricity are present, we are in the case of the (general) **teleparallel geometry**.
- $T^\mu{}_{\nu\rho} = 0$. If the torsion is zero, the connection is said to be *symmetric*, since $\Gamma^\rho{}_{[\mu\nu]} = 0$. This is the case of **Weyl geometry**.
- $Q_{\mu\nu\rho} = 0$. If the non-metricity venishes, the connection is said to be *metric-compatible*, and gives rise to the **Riemann-Cartan geometry**.
- $R^\mu{}_{\nu\rho\sigma} = 0, Q_{\mu\nu\rho} = 0$. If only torsion is non-vanishing, we obtain the case of **metric teleparallel geometry**, or torsional geometry.
- $R^\mu{}_{\nu\rho\sigma} = 0, T^\mu{}_{\nu\rho} = 0$. If only non-metricity is present, we obtain the **symmetric teleparallel geometry**.
- $T^\mu{}_{\nu\rho} = 0, Q_{\mu\nu\rho} = 0$. This is the case the **Riemann geometry**, which describes General Relativity and its enxtensions. In this scenario we have theories with symmetric and metric-compatible connection, which is the Levi-Civita's.
- $R^\mu{}_{\nu\rho\sigma} = 0, T^\mu{}_{\nu\rho} = 0, Q_{\mu\nu\rho} = 0$. If all tensorial quantities are zero, the metric is contrained to be flat, and we are in the case of **Minkowski geometry**.




From now on, in addition to the *over-circled* notation, we will refer to quantities build up on the Levi-Civita connection, (e.g. $\mathring{A}^\mu{}_\nu$), *over-hats* denote quantities related to the teleparallel connection, (e.g. $\hat{A}^\mu{}_\nu$), and *over-diamonds* denote quantities involving non-metricity, (e.g. $\overset{\diamond}{A}{}^\mu{}_\nu$).

## 3.4 Dynamics equivalence

Among all these metric-affine subclasses, the Riemann and Teleparallel ones are particularly relevant. As said before, GR is an example of Riemann theory and is the simplest formulation of gravity which is mediated by the curvature, entirely determined by the metric tensor.
Nevertheless, gravity can also be formulated within alternative geometric frameworks, such as a *pure torsion space*—leading to the *Teleparallel Equivalent of General Relativity (TEGR)* —or a *pure non-metric space*, which gives rise to the *Symmetric Teleparallel Equivalent of General Relativity (STEGR)*. In TEGR, torsion takes over the role of curvature in describing gravitational dynamics, offering an equivalent but conceptually distinct perspective from General Relativity. While GR models gravity through spacetime curvature and identifies geodesics with the trajectories of freely falling particles, TEGR interprets gravity as a gauge force arising from the torsion tensor.
Similarly, STEGR adopts a different geometric approach: both curvature and torsion are set to zero, and the gravitational dynamics are instead governed by the non-metricity tensor. Despite the differences in formulation, STEGR retains several features analogous to TEGR. These three theories consitute the so-called *Geometric Trinity of Gravity*, because, as we will see, they are dynamically equivalent.

GR is formulated in terms of the spacetime metric $g_{\mu\nu}$. In contrast, TEGR uses tetrads $e^A{}_\mu$ to describe the dynamics of gravity, along with a spin connection $\omega^{AB}{}_\mu$, which encodes inertial effects through a flat connection. STEGR, on the other hand, follows the Palatini approach, treating the metric $g_{\mu\nu}$ and the affine connection $\Gamma^\alpha{}_{\mu\nu}$ as independent dynamical entities. Similar to other fundamental interactions in nature, gravity can be reformulated as a gauge theory within both the TEGR and STEGR frameworks.

### 3.4.1 Autoparallels in the Teleparallel framework

We have seen that in GR (see Sec.(2.1)), geodesics and autoparallel curves coincide because the connection is the Levi-Civita. In Metric-Affine theories – and to be precise in the TG and STG formualtion of Gravity – the autoparallel curves are not the same as the geodesics, which means that they are not the extremal curves of the functional length. Hence, in a local chart where $x^\mu$ are the coordinates of $\gamma(\lambda)$, the affinely parametrized autoparallels are defined by

$$\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\lambda^2} + \Gamma^\rho{}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0, \tag{3.4.1}$$

where this time the connection is the generalized one, namely $\Gamma^{\rho}{}_{\mu\nu} = \mathring{\Gamma}^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\mu\nu} + L^{\rho}{}_{\mu\nu}$. Hence, eq.(3.4.1) becomes

$$\boxed{\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\lambda^2} + \mathring{\Gamma}^{\rho}{}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = -\big(K^{\rho}{}_{\mu\nu} + L^{\rho}{}_{\mu\nu}\big)\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}. \tag{3.4.2}$$

Note that this is the general equation for autoparallels in the Teleparallel framework. If we specify it to the case of metric TG, then $L^{\rho}{}_{\mu\nu} = 0$, while if we are considering STG, then $K^{\rho}{}_{\mu\nu} = 0$.

The autoaparallel equation (3.4.2) of a general TG theory reveal a different perspective on GR, no longer viewed solely through the geometric framework of minimal distance paths (geodesics), but instead through a gauge fomulation. Here, the usual geodesic equation is recast into a Lorentz-like force equation[3], where the right-hand side includes a force term involving the contortion tensor, which drives the motion of test particles. This recasting does not alter the physical trajectories – both the geometric and gauge formulations yield the same solutions – but it shifts the interpretation: rather than geometry passively determining the motion of test particles, they are actively influenced by force-like contributions. Specifically, the contortion tensor $K^{\rho}{}_{\mu\nu}$ acts analogously to a Lorentz force, while the disformation tensor $L^{\rho}{}_{\mu\nu}$ contributes a kinetic energy-like effect. This perspective enriches our understanding of GR by emphasizing its dynamical structure, similar to gauge theories in other fundamental interactions.

It is important to note that autoparallels in Teleparallel Gravity are sensitive to parameter changes. Let us start with a curve $\gamma(\lambda)$ which satisfies the autoparallel equation

$$\frac{\mathrm{d}^2 \gamma^\rho}{\mathrm{d}\lambda^2} + \Gamma^{\rho}{}_{\mu\nu}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}\gamma^\nu}{\mathrm{d}\lambda} = 0. \tag{3.4.3}$$

Now, we change the parameter of the curve, namely

$$\lambda = \lambda(\tau). \tag{3.4.4}$$

[3] If a particle has rest mass $m$ and charge $q$ and it is placed in an electromagnetic field described by the Faraday tensor $F_{\mu\nu}$, it satisfies the Lorentz force equation, namely $u^\alpha \nabla_\alpha u^\mu = \frac{q}{m} F^{\mu}{}_{\nu} u^\nu$.

Hence, the trajectory in the spacetime is the same as before, but now it is described by the new parameter $\tau$. Let us compute the first and second derivative:

$$\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\lambda}, \tag{3.4.5}$$

$$\begin{aligned}\frac{\mathrm{d}^2\gamma^\mu}{\mathrm{d}\tau^2} &= \frac{\mathrm{d}^2\lambda}{\mathrm{d}\tau^2}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\lambda} + \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}^2\lambda}{\mathrm{d}\tau^2}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\lambda} + \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}^2\gamma^\mu}{\mathrm{d}\lambda^2} \\ &= \frac{\mathrm{d}^2\lambda}{\mathrm{d}\tau^2}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\lambda} + \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^2\frac{\mathrm{d}^2\gamma^\mu}{\mathrm{d}\lambda^2}. \end{aligned} \tag{3.4.6}$$

If we substitute the second term of the second derivative with the autoparallel equation, we obtain

$$\frac{\mathrm{d}^2\gamma^\mu}{\mathrm{d}\tau^2} = \frac{\mathrm{d}^2\lambda}{\mathrm{d}\tau^2}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\lambda} - \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^2 \Gamma^\mu{}_{\rho\nu}\frac{\mathrm{d}\gamma^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}\gamma^\nu}{\mathrm{d}\lambda}. \tag{3.4.7}$$

We want all the terms to depend explicitly on $\tau$, thus we substitute in eq.(3.4.7)

$$\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\tau}, \tag{3.4.8}$$

and make use of the following manipulation:

$$\frac{\mathrm{d}^2\lambda}{\mathrm{d}\tau^2} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}}\right) = -\frac{1}{\left(\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\right)^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\right) = -\frac{1}{\left(\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\right)^2}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}^2\tau}{\mathrm{d}\lambda^2} = -\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^3\frac{\mathrm{d}^2\tau}{\mathrm{d}\lambda^2}. \tag{3.4.9}$$

It follows that

$$\frac{\mathrm{d}^2\gamma^\mu}{\mathrm{d}\tau^2} + \cancel{\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^2}\Gamma^\mu{}_{\rho\nu}\frac{\cancel{\mathrm{d}\tau}}{\cancel{\mathrm{d}\lambda}}\frac{\mathrm{d}\gamma^\rho}{\mathrm{d}\tau}\frac{\cancel{\mathrm{d}\tau}}{\cancel{\mathrm{d}\lambda}}\frac{\mathrm{d}\gamma^\nu}{\mathrm{d}\tau} = -\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{\cancel{3}\,2}\frac{\mathrm{d}^2\tau}{\mathrm{d}\lambda^2}\frac{\cancel{\mathrm{d}\tau}}{\cancel{\mathrm{d}\lambda}}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\tau},$$

$$\Longrightarrow \boxed{\frac{\mathrm{d}^2\gamma^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\rho\nu}\frac{\mathrm{d}\gamma^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}\gamma^\nu}{\mathrm{d}\tau} = -\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^2\frac{\mathrm{d}^2\tau}{\mathrm{d}\lambda^2}\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\tau}}. \tag{3.4.10}$$

Eq.(3.4.10) tells us that a non-linear change of parametrization, i.e. $\tau \neq a\lambda + b$ with $a, b \in \mathbb{R}$, introduces an extra term in the autoparallel equation. This additional term breaks the autoparallelism of the curve, leading to a curve – parametrized in the new $\tau$ – not autoparallel anymore. This equations quantifies how much the autoparallelism is broken through a term proportional to the second derivative of the change of parametrization.

Furthermore, it is important to note that if we impose the Weitzenböck gauge in TEGR (see later eq.(4.3.9)) and the coincident gauge in STEGR (see later eq. (5.2.8)), we reduce to $\frac{\mathrm{d}^2\gamma^\mu}{\mathrm{d}\tau^2} = 0$, meaning that we are in a LIF.

# Chapter 4

# A possible gauge formulation of Gravity: the Teleparallel Gravity

We now have all the ingredients that form the basis of the theories we want to discuss. In fact, we can provide a gauge formulation of gravity via the Teleparallel Gravity Theory. In this chapter, we will show that this theory can be seen as a translational gauge theory, and then we will analyze two subclasses of this theory, the Metric Teleparallel Gravity and the Symmetric Teleparallel Gravity, which lead to the TEGR and STEGR.

## 4.1 Gauge structure

In many theories of physics, the observable quantities, which are the physical quantities that can be measured, are represented by sections of fiber bundles over some space, as the spacetime. We have seen in Chap.(1) that a fiber bundle together with a continuous group action of a Lie Group give rise to a principal fiber bundle and once a representation of the group is chosen, then we can deal with the associated bundle. Hence, the associated bundle is obtained by replacing the group by one of its linear representations. The elements of a principal bundle can be thought of as generalized frames for the original fiber bundle. Each point of the principal bundle specifies a choice of reference that allows us to convert the intrinsic, coordinate-free description given by a section of the fiber bundle into a concrete representation that we can observe or compute with.
In physics, the generalised frames are called *gauges*, the structure group of the principal bundle is the *gauge group*, while the automorphism of the principal bundle that fixes the base is called *gauge transformation*.

If we consider a vector bundle, the choice of a gauge, that is a local frame of the fiber, allows us to express a section in terms of its components in that frame. As a physical example, we can look at electromagnetism: fixing a gauge corresponds to choosing a local 1-form $A_\mu$ that represents the electromagnetic potential in that frame. Different gauge choices correspond to different potentials related by $A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda$, while the field strength $F_{\mu\nu}$ remains invariant.

Hence, we call *gauge theory* a field theory formulated as the geometry of a principal bundle with Lie group, equipped with a connection (the gauge potential), whose dynamics is determined by an action invariant under local gauge transformations, which are automorphisms of the principal bundle that fix the base manifold and act through the group.

Now, we focus on our theory of interest. Teleparallel Gravity (TG) can be interpreted as a gauge theory of translations [16]. Why translations? We can understand it by invoking the Noether's theorem and recalling that the source of the gravitational field is energy and momentum. In fact, Noether theorem tells us that the energy-momentum current is covariantly conserved once it is ensured that the source Lagrangian is invariant under spacetime translations. If gravitation is represented by a gauge formulation with energy-momentum as the source, then it must be a gauge theory for the translation group.

We want to stress that the conservation of the energy-momentum tensor under spacetime translations does not automatically imply that the theory is a gauge theory for translations. In fact such theory requires the promotion of that (global) symmetry to a local symmetry and the introduction of a gauge field associated to the translational group, which is exactly what we will do in Teleparallel Gravity. For example, QED and Yang-Mills theory are both invariant under global translations, but their gauge structures do not concern the group of translations because this symmetry is not promoted to a local symmetry: QED has

$$\psi \longrightarrow e^{i\alpha(x)}\psi, \; A_\mu \longrightarrow A_\mu - \frac{1}{e}\partial_\mu \alpha(x)$$

as local gauge symmetry transformation, which is based on $U(1)$ group. On the other hand, in Yang-Mills we have

$$A_\mu \longrightarrow U(x) A_\mu U(x)^{-1} + U(x)\partial_\mu U(x)^{-1},$$

where $U(x)$ is an element of the local symmetry gauge group, such as $SU(2)$ or $SU(3)$.
The main difference with TG is that here we want to construct a gauge theory for the spacetime itself, not for an internal symmetry group. Hence, the promotion of this symmetry

to its local version leads to a gauge field for translations, i.e. the tedrad[1]. This is the only method to obtain a gauge field with energy-momentum as source.

### 4.1.1 Geometrical setting

The geometrical setting of Teleparallel Gravity is the tangent bundle $TM$ of a four-dimensional Lorentzian spacetime manifold $(M, g)$. At each point $p \in M$, the tangent space $T_pM$ is a real four-dimensional vector space, which is locally identified with a Minkowski vector space $\mathbb{R}^{1,3}$.

This identification is provided by the tetrad field ${e^a}_\mu$, which acts as a solder form between spacetime and the internal Minkowski space. Pointwise, the tetrad defines an isomorphism

$$ {e^a}_\mu(p) : T_pM \longrightarrow \mathbb{R}^{1,3}. $$

Greek indices label spacetime components, while Latin indices refer to components with respect to a local orthonormal frame in the internal Minkowski space. Local gauge transformations correspond to local Lorentz transformations acting on the internal indices, and represent changes of local inertial frames rather than transformations of spacetime coordinates.

Hence, the spacetime $M$ plays the role of the base manifold, while the tangent spaces $T_pM$ are the fibers. Via the tetrad field, each fiber is locally identified with an internal Minkowski space, where the gauge transformations takes place.

### 4.1.2 Gauge invariance

The generators of such transformations form the translation Lie group $T^4$, and are differential operators $P_a = \frac{\partial}{\partial x^a} = \partial_a$. Translations commute, thus the satisfy the commutation relation $[\partial_a, \partial_b] = 0$.
Let us define the gauge transformation as

$$ x^a \longrightarrow \bar{x}^a = x^a + \epsilon^a(x^\mu), \tag{4.1.1} $$

[1]Actually, we will see that it is not really the tetrad itself, but the tetrad is constructed such that it contains the true gauge potential.

where $\epsilon^a(x^\mu)$ is the infinitesimal transformation parameter. Transformation (4.1.1) can be written in terms of the generators by means of an expansion:

$$\delta \bar{x}^a = \bar{x}^a - x^a = \epsilon^b(x^\mu)\frac{\partial x^a}{\partial x^b} = \epsilon^a(x^\mu). \tag{4.1.2}$$

Let us talk about source fields. As it has been said in Chap.(1), fields are (local) sections of a fiber bundle, thus it is a map that takes a point $x^\mu$ and gives an object $\Psi(x^a(x^\mu))$ back, which is the field[2]. Such field transforms under (4.1.1) according to the total variation

$$\begin{aligned}\delta_\epsilon \Psi(x^a(x^\mu)) &= \Psi(x^a(x^\mu) + \epsilon^a(x^\mu)) - \Psi(x^a(x^\mu)) \\ &= \Psi(x^a(x^\mu)) + \epsilon^a(x^\mu)\partial_a \Psi(x^a(x^\mu)) - \Psi(x^a(x^\mu)) \\ &= \epsilon^a(x^\mu)\partial_a \Psi(x^a(x^\mu)).\end{aligned} \tag{4.1.3}$$

This gives the change of $\Psi$ at a fixed $x^a$ and at fixed spacetime point $x^\mu$.

If the gauge is *global*, meaning that the parameter of the transformation does not depend on the point of the spacetime, i.e. $e^a$ = constant, then there are no additional terms once an ordinary derivative $\partial_\mu \Psi$ is performed, thus it trasforms covariantly:

$$\delta_\epsilon(\partial_\mu \Psi) = \partial_\mu(\delta_\epsilon \Psi) = \partial_\mu(\epsilon^a \partial_a \Psi) = \epsilon^a \partial_a \partial_\mu \Psi. \tag{4.1.4}$$

For a *local* transformation with parameter $\epsilon^a(x^\mu)$, instead, the field $\partial_\mu \Psi$ does not transform covariantly:

$$\delta_\epsilon(\partial_\mu \Psi) = \partial_\mu(\delta_\epsilon \Psi) = \partial_\mu(\epsilon^a(x^\mu)\partial_a \Psi) = \epsilon^a(x^\mu)\partial_a\partial_\mu \Psi + (\partial_\mu \epsilon^a(x^\mu))\partial_a \Psi \neq \epsilon^a(x^\mu)\partial_a\partial_\mu \Psi, \tag{4.1.5}$$

where the additional derivative term, called sporious term, breaks the translational gauge covariance of the tranformation. To restore the covariance, we will follow the standard procedure of all the gauge theories, previously introduced in Chap.(1.10): we will introduce a *translational gauge potential* $B_\mu$, which is a 1-form assuming values inthe Lie algebra of the translational group, thus it can be written in terms of its generators:

$$B_\mu = B^a{}_\mu \partial_a. \tag{4.1.6}$$

[2] $x^a(x^\mu)$ is a point on the fiber over $x^\mu$

This potential could be used to construct the *gauge potential derivative*, that is, using trivial tetrads,

$$\boxed{e'_\mu\Psi = \partial_\mu\Psi + B^a{}_\mu\partial_a\Psi = e'^a{}_\mu\partial_a\Psi}, \tag{4.1.7}$$

which holds in the class of Lorentz inertial frames in which there are no inertial effects. To make the covariant derivative transform covariantly, we require that the gauge potential transforms in such a way that cancels the sporious term out:

$$\boxed{\delta_\epsilon B^a{}_\mu = -\partial_\mu\epsilon^a(x^\mu)}. \tag{4.1.8}$$

In this way, $e'_\mu\Psi$ transforms covariantly under gauge translations:

$$\begin{aligned}
\delta_\epsilon(e'_\mu\Psi) &= \delta_\epsilon(\partial_\mu\Psi + B^a{}_\mu\partial_a\Psi)\\
&= \delta_\epsilon(\partial_\mu\Psi) + \delta_\epsilon(B^c{}_\mu\partial_c\Psi) \qquad \text{(using (4.1.7))}\\
&= \epsilon^a(x^\mu)\partial_a\partial_\mu\Psi + (\partial_\mu\epsilon^a(x^\mu))\partial_a\Psi + \delta_\epsilon B^c{}_\mu\partial_c\Psi + B^c{}_\mu\partial_c\delta_\epsilon\Psi \qquad \text{(using (4.1.5))}\\
&= \epsilon^a(x^\mu)\partial_a\partial_\mu\Psi + (\partial_\mu\epsilon^a(x^\mu))\partial_a\Psi + \delta_\epsilon B^c{}_\mu\partial_c\Psi + B^c{}_\mu\partial_c(\epsilon^a(x^\mu)\partial_a\Psi) \qquad \text{(using (4.1.3))}\\
&= \epsilon^a(x^\mu)\partial_a(\partial_\mu\Psi + B^c{}_\mu\partial_c\Psi) + \cancel{(\partial_\mu\epsilon^a(x^\mu))\partial_a\Psi} - \cancel{(\partial_\mu\epsilon^a(x^\mu))\partial_a\Psi} \qquad \text{(using (4.1.8))}\\
&= e^a(x^\mu)\partial_a(e'_\mu\Psi)
\end{aligned} \tag{4.1.9}$$

Since teleparallel gravity is constructed using tetrad fields, which provide a relationship between space (internal) tensors and spacetime (external) tensors, the gauge covariant derivative can be written in terms of a general non-trivial tetrad $e_\mu$:

$$\boxed{e_\mu\Psi = \partial_\mu x^a\partial_a\Psi + B^a{}_\mu e^a{}_\mu\partial_a\Psi = e^a{}_\mu\partial_a\Psi}, \tag{4.1.10}$$

where

$$e^a{}_\mu = \partial_\mu x^a + B^a{}_\mu \tag{4.1.11}$$

is the non-trivial tetrad, meaning that $B^a{}_\mu \neq \partial_\mu\epsilon^a(x^\mu)$, otherwise it would be just a translational gauge transformation of the trivial tetrad $e^a{}_\mu = \partial_\mu x^a$.

This formalism is replicable in a general Lorentz frame and can be obtained by performing a local Lorentz transformation (1.16.1) on the gauge potential $B^a{}_\mu$, assuming that the latter transforms as a Lorentz vector in the algebraic index:

$$B^a{}_\mu \longrightarrow \Lambda^a{}_b(x) B^b{}_\mu. \tag{4.1.12}$$

Thus, the covariant derivative (4.1.7) admits the following generalization:

$$\begin{aligned} e_\mu \Psi \stackrel{\Lambda}{\longrightarrow} e_\mu \Psi &= (\Lambda^a{}_b e^b{}_\mu)(\Lambda^c{}_c \partial_c \Psi) \\ &= [\partial_\mu (\Lambda^a{}_b x^b) + \Lambda^a{}_b B^b{}_\mu] \Lambda^c{}_a \partial_c \Psi \\ &= [(\partial_\mu \Lambda^a{}_b) x^b + \Lambda^a{}_b \partial_\mu x^b + \Lambda^a{}_b B^b{}_\mu] \Lambda^c{}_a \partial_c \Psi \\ &= (\partial_\mu \Lambda^a{}_b) x^b \Lambda^c{}_a \partial_c \Psi + \Lambda^a{}_b \partial_\mu x^b \Lambda^c{}_a \partial_c \Psi + \Lambda^a{}_b B^b{}_\mu \Lambda^c{}_a \partial_c \Psi \\ &= \Lambda^c{}_a \partial_\mu \Lambda^a{}_b x^b \partial_c \Psi + \partial_\mu x^c \partial_c \Psi + B^c{}_\mu \partial_c \Psi \\ &= (\partial_\mu x^a + \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b + B^a{}_\mu) \partial_a \Psi, \end{aligned} \tag{4.1.13}$$

where $\overset{\bullet}{\omega}{}^a{}_{b\mu}$ is the purely inertial connection (1.16.6), $\overset{\bullet}{\omega}{}^a{}_{b\mu} = \Lambda^a{}_c(x) \partial_\mu \Lambda^c{}_b(x)$.

$$\boxed{e_\mu \Psi = e^a{}_\mu \partial_a \Psi = (\partial_\mu x^a + \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b + B^a{}_\mu) \partial_a \Psi}, \tag{4.1.14}$$

where the tetrad components are now

$$e^a{}_\mu = \partial_\mu x^a + \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b + B^a{}_\mu. \tag{4.1.15}$$

As we see, the tetrad components contain the trivial tetrad (1.16.5), i.e. $e^a{}_\mu = \partial_\mu x^a + \overset{\bullet}{\omega}{}^a_{b\mu} x^b = \overset{\bullet}{\mathcal{D}}_\mu x^a$, hence they can be rewritten as

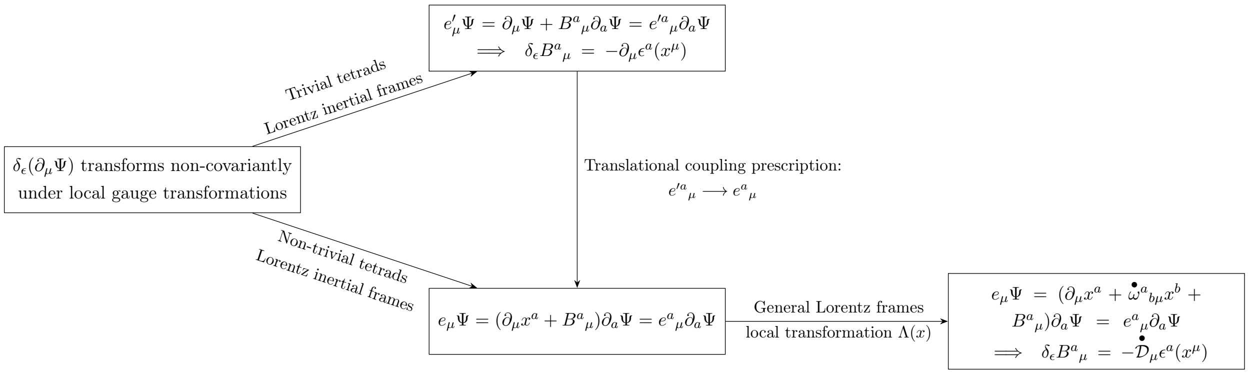
$$e^a{}_\mu = \overset{\bullet}{\mathcal{D}}_\mu x^a + B^a{}_\mu. \tag{4.1.16}$$

This means that for general non-trivial tetrads, the gauge transformation of $B^a{}_\mu$ (4.1.8) has to be upgraded to

$$\boxed{\delta_\epsilon B^a{}_\mu = -\overset{\bullet}{\mathcal{D}}_\mu \epsilon^a(x^\mu)}. \tag{4.1.17}$$

Note that in the class of frames in which the inertial spin connection $\overset{\bullet}{\omega}{}^a{}_{b\mu}$ vanishes, it assumes the form (4.1.8).

We can summarize what we have done with the following diagram and formulating the *gravitational coupling prescription in Teleparallel Gravity.*




## 4.2 Gravitational coupling prescription

We want to interpret the transformations of the gauge potential made in the previous section as a gravitational coupling prescription for a Teleparallel Gravity theory. To do so, we have to take into account the following two coupling prescriptions:

### 4.2.1 Translational coupling prescription

The *translational coupling prescription*, i.e. $e'^a{}_\mu \longrightarrow e^a{}_\mu$, allows us to switch gravitational effects on, retrieving the covariant derivative in a non-trivial tetrad (see eq.(4.1.10)). The translational coupling prescription naturally leads to the promotion of the spacetime Minkowski metric to a general Riemannian metric, which is the so-called *gravitational coupling prescription*:

$$\eta_{\mu\nu} = \eta_{ab}e'^a{}_\mu e'^b{}_\nu \longrightarrow g_{\mu\nu} = \eta_{ab}e^a{}_\mu e^b{}_\nu. \tag{4.2.1}$$

It is important to note that in General Relativity this replacement is implicitly assumed whenever we apply the gravitational coupling prescription. Instead, in Teleparallel Gravity, it is a consequence of the translational coupling prescription application.

### 4.2.2 Lorentz coupling prescription

Since Lorentz invariance is a fundamental symmetry of nature, Teleparallel Gravity must also be invariant under local Lorentz transformations. This requirement introduces the so-called *Lorentz coupling prescription*, which arises naturally as a consequence of the *Strong Equivalence Principle* (see Chap.(2)) and provides an additional correction term in the derivative. In fact, this prescription ultimately stems from the General Covariance Principle (GCP), which can be interpreted as the active version of the Strong Equivalence Principle. It states that any equation valid in a gravitational field must reduce locally – that is, at a point or along a geodesic – to its special-relativistic counterpart.

The first step to obatin the Lorentz coupling prescription is to move to a general anholonomic frame. Let us consider a vector field $\Phi'^c$ in special relativity, i.e. in Minkowsi spacetime. In the trivial holonomic frame,

$$e'_a = \delta_a{}^\mu \partial_\mu \implies e'_a \Phi'^c = \delta_a{}^\mu \partial_\mu \Phi'^c. \tag{4.2.2}$$

Under a local Lorentz transformation,

$$\Phi'^c \longrightarrow \Phi^c = \Lambda^c{}_d \Phi'^d, \tag{4.2.3}$$

then we have that

$$\begin{aligned}
e'_a \Phi'^c &= \delta_a{}^\mu \partial_\mu (\Lambda_d{}^c \Phi^d) \\
&= \Lambda^b{}_a \Lambda_d{}^c \partial_b \Phi^d + \Lambda^b{}_a (\partial_b \Lambda_d{}^c) \Phi^d \\
&= \Lambda^b{}_a \Lambda_d{}^c \partial_b \Phi^d + \Lambda^b{}_a \Lambda_e{}^c \overset{\bullet}{\omega}{}^e{}_{bd} \Phi^d \qquad \text{(inverting eq.(1.16.6))} \\
&= \Lambda^b{}_a \Lambda_d{}^c \partial_b \Phi^d + \Lambda^b{}_a \Lambda_d{}^c \overset{\bullet}{\omega}{}^d{}_{be} \Phi^e \qquad \text{(renaming } d \leftrightarrow e\text{)} \\
&= \Lambda^b{}_a \Lambda_d{}^c (\partial_b \Phi^d + \overset{\bullet}{\omega}{}^d{}_{be} \Phi^e) \\
&= \Lambda^b{}_a \Lambda_d{}^c \big(\partial_b \Phi^d + \frac{1}{2}(f_e{}^d{}_b + f_b{}^d{}_e - f^d{}_{eb}) S^e{}_f \Phi^f\big), \qquad \text{(using eq.(1.16.12))}
\end{aligned} \tag{4.2.4}$$

where we write the field in terms of the vector representation of the Lorentz generators.

For a scalar field $\Psi$, making use of the GCP, perform a local Lorentz transformation, such that all derivatives $\partial_\mu \Psi$ assume the Lorentz covariant form, as done before for $\Phi$:

$$\partial_\mu \Psi \longrightarrow \mathcal{D}_\mu \Psi = \partial_\mu + \frac{1}{2} e'^a{}_\mu (f'^a_b{}_c + f'^c_a{}_b - f'^c{}_{ab}) S_c{}^b \Psi, \tag{4.2.5}$$

where $f^c{}_{ab}$ are now the anholonomy coefficients of the trivial tetrad in the Minkowski spacetime and $S_c{}^b$ are the Lorentz generators in the same representation to which $\Psi$ belongs. The second term is the one which mantains the local Lorentz covariance of the derivative in the new inertial frame.

Now, in the presence of gravitation, according to the translational coupling prescription $e'^a{}_\mu \longrightarrow e^a{}_\mu$, the trivial tedrad of eq.(4.2.5) is substituted for a non-trivial one $e^a{}_\mu$, and we obtain the final gravitational coupling prescription for Teleparallel Gravity:

$$\boxed{\partial_\mu \Psi \longrightarrow \mathcal{D}_\mu \Psi = \partial_\mu + \frac{1}{2} e^a{}_\mu (f_b{}^a{}_c + f_a{}^c{}_b - f^c{}_{ab}) S_c{}^b \Psi} \tag{4.2.6}$$

with anholonomy coefficients (1.9.10)

$$f^c{}_{ab} = e_a{}^\mu e_b{}^\nu (\partial_\nu e^c{}_\mu - \partial_\mu e^c{}_\nu). \tag{4.2.7}$$

We can then compare the gravitational coupling prescription of TG and GR as follows:

| TG | | GR |
|---|---|---|
| $e'^a{}_\mu \longrightarrow e^a{}_\mu$ | $\longleftrightarrow$ | $\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$ |
| $\partial_\mu \longrightarrow \mathcal{D}_\mu$ | | |

## 4.3 Metric Teleparallel Gravity

We want now to explore the specific case in which the theory has $R^\mu{}_{\nu\rho\sigma} = Q_{\rho\mu\nu} = 0$. This means that the theory is metric-compatible and the torsion is the only geometrical object that describes the gravitational dynamics. It is also called Torsional Teleparallel Gravity, or simply Teleparallel Gravity (TG).

We have seen that this theory is a gauge-translational formulation of gravity, and as in any gauge theory, the field strength of the (Metric) Teleparallel Gravity can be obtained from the commutation relation of gauge covariant derivatives. Using the translational covariant derivative (4.1.14), $e_\mu \Psi = (\partial_\mu + \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b \partial_a + B^a{}_\mu \partial_a)\Psi$ we have

$$[e_\mu, e_\nu] = \hat{T}^a{}_{\mu\nu} \partial_a, \tag{4.3.1}$$

where

$$\hat{T}^a{}_{\mu\nu} = \partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu + \overset{\bullet}{\omega}{}^a{}_{b\mu} B^b{}_\nu - \overset{\bullet}{\omega}{}^a{}_{b\nu} B^b{}_\mu = \overset{\bullet}{\mathcal{D}}_\nu B^a{}_\nu - \overset{\bullet}{\mathcal{D}}_\nu B^a{}_\mu, \tag{4.3.2}$$

is the translational field strength. We can add the term

$$[\overset{\bullet}{\mathcal{D}}_\mu, \overset{\bullet}{\mathcal{D}}_\nu] x^a = \overset{\bullet}{\mathcal{D}}_\mu (\overset{\bullet}{\mathcal{D}}_\nu x^a) - \overset{\bullet}{\mathcal{D}}_\nu (\overset{\bullet}{\mathcal{D}}_\mu x^a) = 0 \tag{4.3.3}$$

and then rewrite eq.(4.3.2) as

$$\hat{T}^{a}{}_{\mu\nu} = \overset{\bullet}{\mathcal{D}}_{\mu} e^{a}{}_{\nu} - \overset{\bullet}{\mathcal{D}}_{\nu} e^{a}{}_{\mu} = \partial_{\mu} e^{a}{}_{\nu} - \partial_{\nu} e^{a}{}_{\mu} + \overset{\bullet}{\omega}{}^{a}{}_{b\mu} e^{b}{}_{\nu} - \overset{\bullet}{\omega}{}^{a}{}_{b\nu} e^{b}{}_{\mu}. \tag{4.3.4}$$

Moreover, through a contraction with a tetrad, we can retrieve the usual form of the torsion (1.7.2):

$$\begin{aligned} \hat{T}^{\rho}{}_{\mu\nu} &= e_{a}{}^{\rho} \hat{T}^{a}{}_{\mu\nu} \\ &= e_{a}{}^{\rho} \partial_{\mu} e^{a}{}_{\nu} - e_{a}{}^{\rho} \partial_{\nu} e^{a}{}_{\mu} + e_{a}{}^{\rho} \overset{\bullet}{\omega}{}^{a}{}_{b\mu} e^{b}{}_{\nu} - e_{a}{}^{\rho} \overset{\bullet}{\omega}{}^{a}{}_{b\nu} e^{b}{}_{\mu} && (4.3.5) \\ &= \Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu}, && (4.3.6) \end{aligned}$$

where

$$\Gamma^{\rho}{}_{\nu\mu} = e_{a}{}^{\rho} \partial_{\mu} e^{a}{}_{\nu} + e_{a}{}^{\rho} \overset{\bullet}{\omega}{}^{a}{}_{b\mu} e^{b}{}_{\nu} \tag{4.3.7}$$

is the *Weitzenböck connection*, which is the *non-trivial spacetime-indexed connection* corresponding to the inertial spin conncection $\overset{\bullet}{\omega}{}^{a}{}_{b\mu}$. It can also be written in the form

$$\partial_{\mu} e^{a}{}_{\nu} + \overset{\bullet}{\omega}{}^{a}{}_{b\mu} e^{b}{}_{\nu} - \Gamma^{\rho}{}_{\nu\mu} e^{a}{}_{\rho} = 0. \tag{4.3.8}$$

In the class of frames in which the pure inertial spin connection $\overset{\bullet}{\omega}{}^{a}{}_{b\mu}$ vanishes, the Weitzenböck connection reduces to

$$\partial_{\mu} e^{a}{}_{\nu} - \Gamma^{\rho}{}_{\nu\mu} e^{a}{}_{\rho} = 0, \tag{4.3.9}$$

giving rise to the *Weitzenböck gauge.* In this gauge, torsion (4.3.5) assumes the form

$$\hat{T}^{\rho}{}_{\mu\nu} = e_{a}{}^{\rho} \partial_{\mu} e^{a}{}_{\nu} - e_{a}{}^{\rho} \partial_{\nu} e^{a}{}_{\mu}. \tag{4.3.10}$$

As we said in Sec.(1.9), the spin connection is linked to the inertial effects present in the tetrad frame, and it is covariant under both diffeomorphism and local Lorentz transformation. Thus, since the torsion is written in terms of spin connection, then it inherits the same properties.

### 4.3.1 Gravitation coupling prescription in (metric) Teleparallel Gravity

A straightforward consequence of the non-vanishing torsion is that it is possible to rewrite the anholonomy coefficients (4.2.7) using spin connections. From eq.(4.3.4),

$$\hat{T}^a{}_{\nu\mu} + \overset{\bullet}{\omega}{}^a{}_{b\nu} e^b{}_\mu - \overset{\bullet}{\omega}{}^a{}_{b\mu} e^b{}_\nu = \partial_\nu e^a{}_\mu - \partial_\mu e^a{}_\nu, \tag{4.3.11}$$

and than we have

$$f^c{}_{ab} = e_a{}^\mu e_b{}^\nu (\partial_\nu e^c{}_\mu - \partial_\mu e^c{}_\nu) = e_a{}^\mu e_b{}^\nu (\hat{T}^c{}_{\nu\mu} - \overset{\bullet}{\omega}{}^c{}_{d\nu} e^d{}_\mu + \overset{\bullet}{\omega}{}^c{}_{d\mu} e^d{}_\nu), \tag{4.3.12}$$

which can be written as

$$\boxed{f^c{}_{ab} - \hat{T}^c{}_{ba} = \overset{\bullet}{\omega}{}^c{}_{ba} - \overset{\bullet}{\omega}{}^c{}_{ab}}. \tag{4.3.13}$$

This equation is remarkable because it exploits the form of the anholonomy coefficients in presence of torsion (cf. eq.(1.16.10), $f^c{}_{ab} = \overset{\bullet}{\omega}{}^c{}_{ba} - \overset{\bullet}{\omega}{}^c{}_{ab}$). This expression can be recasted as follows:

$$\frac{1}{2}(f_b{}^c{}_a + f_a{}^c{}_b - f^c{}_{ba}) = \overset{\bullet}{\omega}{}^c{}_{ba} + \hat{K}^c{}_{ba}, \tag{4.3.14}$$

where

$$\hat{K}^c{}_{ba} = \frac{1}{2}(\hat{T}_b{}^c{}_a + \hat{T}_a{}^c{}_b - \hat{T}^c{}_{ba}). \tag{4.3.15}$$

Thus, since the coupling prescription found previously was (4.2.6)

$$\partial_\mu \Psi \longrightarrow \mathcal{D}_\mu \Psi = \partial_\mu + \frac{1}{2} e^a{}_\mu (f_b{}^a{}_c + f_a{}^c{}_b - f^c{}_{ab}) S_c{}^b \Psi, \tag{4.3.16}$$

it can be further developed as

$$\boxed{\partial_\mu \Psi \longrightarrow \mathcal{D}_\mu \Psi = \partial_\mu + \frac{1}{2}(\overset{\bullet}{\omega}{}^c{}_{b\mu} + \hat{K}^c{}_{b\mu}) S_c{}^b \Psi}. \tag{4.3.17}$$

In GR, the spin connection takes the form of the full expression of the coefficients of the anholonomy in eq.(4.3.14) as

$$\mathring{\omega}^{c}{}_{ab} = \frac{1}{2}(f_b{}^c{}_a + f_a{}^c{}_b - f^c{}_{ba}), \tag{4.3.18}$$

which is the *Levi-Civita (torsionless) spin connection* and results, thanks to the fundamental identity of the Lorentz connection [13][17], in

$$\boxed{\overset{\bullet}{\omega}{}^{c}{}_{b\mu} - \hat{K}^{c}{}_{b\mu} = \mathring{\omega}^{c}{}_{b\mu}}, \tag{4.3.19}$$

where GR and TG are joined together. Since the gravitational coupling prescription for GR is

$$\partial_\mu \Psi \longrightarrow \mathcal{D}_\mu \Psi = \partial_\mu \Psi + \frac{1}{2}\mathring{\omega}^{c}{}_{b\mu} S_c{}^b \Psi, \tag{4.3.20}$$

the gravitational coupling prescription for TG is found to be equivalent of GR's one. Since both prescription were obtained from the general covariance principle, both are consistent with the Strong Equivalence Principle.

It is important to note that the GR spin connection $\mathring{\omega}^{c}{}_{b\mu}$ provides both gravitational and inertial effects. Instead in TG, $\overset{\bullet}{\omega}{}^{c}{}_{b\mu}$ describes only the inertial effects, i.e. those depending on the choice of the frame, while $\hat{K}^{c}{}_{b\mu}$ takes into account the gravitation effect, which is frame independent. In this view, the (Strong) Equivalence Principle acquires a deeper and more tangible interpretation: unlike in GR, TG allows a clear separation between inertial and gravitational effects, thereby clarifying their physical roles and operational meaning.

Furthermore, in a local frame in which the GR spin connection vanishes, i.e. $\mathring{\omega}^{c}{}_{b\mu} = 0$, eq.(4.3.19) becomes

$$\overset{\bullet}{\omega}{}^{c}{}_{b\mu} = \hat{K}^{c}{}_{b\mu}, \tag{4.3.21}$$

which states that in this local frame the inertial effects are compensating those of gravitation.

Another remarkable property of eq.(4.3.19) is that, although the purely inertial spin connection $\overset{\bullet}{\omega}{}^{a}{}_{b\mu}$ is more general than the GR spin connection, the difference between $\overset{\bullet}{\omega}$ and the contortion tensor $\hat{K}$ is equivalent to the GR spin connection. This means that the gravitational coupling prescription (4.3.17) does not introduce any additional degrees of freedom in relation to GR. This description is then consistent with those in which the source is the ten-components symmetric energy-momentum tensor.

### 4.3.2 Tetrad and its associated spin connection

Let us start recalling that in GR the fundamental dynamical object is the metric tensor $g_{\mu\nu}$, while in TG the metric becomes a derived quantity, and its role is taken over by the tetrad $e^a{}_\mu$ and the spin connection $\omega^a{}_{b\mu}$, which serve as the fundamental variables. Thus, tetrad $e^a{}_\mu$ is associated with an inertial spin connection $\overset{\bullet}{\omega}{}^a{}_{b\mu}$ that describes the inertial effects in such a frame, providing the pair $\{e^a{}_\mu, \overset{\bullet}{\omega}{}^a{}_{b\mu}\}$. We know that there is a class of frames, the proper frames, in which the spin connection vanishes, i.e. $\{e^a{}_\mu, 0\}$. A general class of frames is obtained by a local Lorentz transformation on the proper frames, thus we have an infinity of $\{e^a{}_\mu, \overset{\bullet}{\omega}{}^a{}_{b\mu}\}$ pairs with non-vanishing spin connection.
We now wonder how we can associate to a tetrad its spin connection. The method we will propose is based on determining the inertial effects present in the frame $e^a{}_\mu$ and then determining the spin connection that compensates for those effects [24].
Let us start defining the *reference tetrad*, $e^a_{(r)\mu}$, as a tetrad in which gravity is switched off:

$$e^a_{(r)\mu} = \lim_{G\to 0} e^a{}_\mu. \tag{4.3.22}$$

In a reference tetrad, the gravitational potential $B^a{}_\mu$ does not appear and it can be written as (see eq.(4.1.15))

$$e^a_{(r)\mu} = \partial_\mu x^a + \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b. \tag{4.3.23}$$

A reference tetrad reduces to a trivial tetrad, since with zero gravity the space becomes flat and we have (1.9.6), and the torsion tensor associated with that spin connection vanishes:

$$\hat{T}^a{}_{bc}(e^a_{(r)\mu}, \overset{\bullet}{\omega}{}^a{}_{b\mu}) = 0. \tag{4.3.24}$$

Hence, using eq.(4.3.13), it follows that

$$\hat{T}^a{}_{bc}(e^a_{(r)\mu}, \overset{\bullet}{\omega}{}^a{}_{b\mu}) = \overset{\bullet}{\omega}{}^a{}_{cb} - \overset{\bullet}{\omega}{}^a{}_{bc} - f^a{}_{bc}(e^a_{(r)\mu}) = 0, \tag{4.3.25}$$

and following the same procedure as for (1.16.12), we obtain

$$\overset{\bullet}{\omega}{}^a{}_{bc} = \frac{1}{2}\left[ f_b{}^a{}_c(e^a_{(r)\mu}) + f_c{}^a{}_b(e^a_{(r)\mu}) - f^a{}_{bc}(e^a_{(r)\mu}) \right], \tag{4.3.26}$$

which is the inertial spin connection naturally associated to the reference tetrad $e^a_{(r)\mu}$. Now, since the reference tetrad $e^a_{(r)\mu}$ and the usual tetrad $e^a{}_\mu$ only differ by their gravitational content because the inertial one is the same, then the inertial spin connection (4.3.26) is the same inertial spin connection naturally associated with the usual tetrad $e^a{}_\mu$.

### 4.3.3 Teleparallel Equivalent of General Relativity (TEGR)

We want now show that (Metric) Teleparallel Gravity (TG) with a particular Lagrangian is dynamically equivalent to that of GR, providing the so-called *Teleparallel Equivalent of General Relativity.* The main focus of this section is therefore finding the Ricci scalar in TG setting, since we know its role in the GR action is central (see Sec.(2.2)).

We have seen that the generalized affine connection admits the following decomposition (3.1.7)

$$\Gamma^\alpha{}_{\mu\nu} := \mathring{\Gamma}^\alpha{}_{\mu\nu} + K^\alpha{}_{\mu\nu} + L^\alpha{}_{\mu\nu}. \tag{4.3.27}$$

It is fundamental to recall the expression of the curvature tensor (1.7.4) in terms of the generalized affine connection:

$$R^\alpha{}_{\beta\mu\nu} = \mathring{R}^\alpha{}_{\beta\mu\nu} + \mathring{\nabla}_\mu D^\alpha{}_{\nu\beta} - \mathring{\nabla}_\nu D^\alpha{}_{\mu\beta} + D^\sigma{}_{\nu\beta} D^\alpha{}_{\mu\sigma} - D^\sigma{}_{\mu\beta} D^\alpha{}_{\nu\sigma}, \tag{4.3.28}$$

with

$$D^\alpha{}_{\mu\nu} = K^\alpha{}_{\mu\nu} + L^\alpha{}_{\mu\nu}. \tag{4.3.29}$$

In the case of Torsional TG the curvature vanishes, then $L^\rho{}_{\mu\nu} = 0$. Hence, in this setting the TG affine connection is

$$\boxed{\hat{\Gamma}^\alpha{}_{\mu\nu} = \mathring{\Gamma}^\alpha{}_{\mu\nu} + \hat{K}^\alpha{}_{\mu\nu}}, \tag{4.3.30}$$

while the curvature tensor (4.3.28) becomes

$$R^\alpha{}_{\beta\mu\nu} = \mathring{R}^\alpha{}_{\beta\mu\nu} + \mathring{\nabla}_\mu \hat{K}^\alpha{}_{\nu\beta} - \mathring{\nabla}_\nu \hat{K}^\alpha{}_{\mu\beta} + \hat{K}^\sigma{}_{\nu\beta} \hat{K}^\alpha{}_{\mu\sigma} - \hat{K}^\sigma{}_{\mu\beta} \hat{K}^\alpha{}_{\nu\sigma} = 0, \tag{4.3.31}$$

where the vanishing property of the curvture of TG has been imposed. To obtain the Ricci scalar, we have to contract the first with the third index, namely $R_{\beta\nu} = R^\alpha{}_{\beta\alpha\nu}$, and then contract the remaining indeces with the metric, $R = g^{\beta\nu} R_{\beta\nu}$. It follows that

$$\mathring{R} + \mathring{\nabla}_\mu \hat{K}^{\mu\nu}{}_\nu - \mathring{\nabla}_\nu \hat{K}_\mu{}^{\mu\nu} + \hat{K}^{\sigma\nu}{}_\nu \hat{K}^\mu{}_{\mu\sigma} - \hat{K}^\sigma{}_{\mu\nu} \hat{K}^{\mu\nu}{}_\sigma = 0. \tag{4.3.32}$$

It is possible to manipulate the terms of this equation:

$$\begin{aligned}
\mathring{\nabla}_\mu \hat{K}^{\mu\nu}{}_\nu - \mathring{\nabla}_\nu \hat{K}_\mu{}^{\mu\nu} &= \mathring{\nabla}_\mu (\hat{K}^{\mu\nu}{}_\nu - \hat{K}_\nu{}^{\nu\mu}) \\
&= 2\mathring{\nabla}_\mu \hat{K}^{\mu\nu}{}_\nu \qquad \text{(using (3.1.10): } K^\rho{}_{\mu\nu} = -K_{\nu\mu}{}^\rho) \\
&= 2\mathring{\nabla}\Big[\frac{1}{2}(\hat{T}^{\mu\nu}{}_\nu + \hat{T}^{\nu\mu}{}_\nu + \hat{T}_\nu{}^{\mu\nu})\Big] \qquad \text{(using (3.1.8))} \\
&= 2\mathring{\nabla}_\mu \Big(\frac{1}{2} 2\hat{T}_\nu{}^{\mu\nu}\Big) = 2\mathring{\nabla}_\mu \hat{T}^\mu,
\end{aligned} \tag{4.3.33}$$

where $\hat{T}_\nu{}^{\mu\nu} = \hat{T}^\mu$ is called *torsion tensor*, while the first term $\hat{T}^{\mu\nu}{}_\nu = 0$ because the torsion is antysimmetric (1.7.7) and contractions on antisymmetric indeces give zero. Then we have the term

$$\hat{K}^{\sigma\nu}{}_\nu \hat{K}^\mu{}_{\mu\sigma} = -\hat{K}^{\sigma\nu}{}_\nu \hat{K}_{\sigma\mu}{}^\mu = -\hat{T}^\alpha \hat{T}_\alpha, \tag{4.3.34}$$

because, as in the previous calculation, we have torsion terms with antisymmetric indeces contracted. Finally, the last term, which instead does not show this type of contraction, becomes:

$$\begin{aligned}
\hat{K}^\sigma{}_{\mu\nu} \hat{K}^{\mu\nu}{}_\sigma &= \hat{K}_{\sigma\mu\nu} \hat{K}^{\sigma\nu\mu} \\
&= \frac{1}{4}\big(\hat{T}_{\sigma\mu\nu} + \hat{T}_{\mu\sigma\nu} + \hat{T}_{\nu\sigma\mu}\big)\big(\hat{T}^{\sigma\nu\mu} + +\hat{T}^{\nu\sigma\mu} + \hat{T}^{\mu\sigma\nu}\big) \\
&= \frac{1}{4}\big(\hat{T}_{\sigma\mu\nu}\hat{T}^{\sigma\nu\mu} + \hat{T}_{\sigma\mu\nu}\hat{T}^{\nu\sigma\mu} + \hat{T}_{\sigma\mu\nu}\hat{T}^{\mu\sigma\nu} + \hat{T}_{\mu\sigma\nu}\hat{T}^{\sigma\nu\mu} \\
&\qquad + \hat{T}_{\mu\sigma\nu}\hat{T}^{\nu\sigma\mu} + \hat{T}_{\mu\sigma\nu}\hat{T}^{\mu\sigma\nu} + \hat{T}_{\nu\sigma\mu}\hat{T}^{\sigma\nu\mu} + \hat{T}_{\nu\sigma\mu}\hat{T}^{\nu\sigma\mu} + \hat{T}_{\nu\sigma\mu}\hat{T}^{\mu\sigma\nu}\big) \\
&= \frac{1}{4}\big(\hat{T}^{\alpha\mu\nu}\hat{T}_{\alpha\mu\nu} + 2\hat{T}_{\alpha\mu\nu}\hat{T}^{\mu\alpha\nu}\big).
\end{aligned} \tag{4.3.35}$$

Using eqs.(4.3.33), (3.1.5) and (4.3.35) we can rewrite the Ricci scalar (4.3.32):

$$\mathring{R} = \frac{1}{4}\big(-\hat{T}^{\alpha\mu\nu}\hat{T}_{\alpha\mu\nu} - 2\hat{T}_{\alpha\mu\nu}\hat{T}^{\mu\alpha\nu}\big) + \hat{T}^\alpha\hat{T}_\alpha - 2\mathring{\nabla}_\mu\hat{T}^\mu, \tag{4.3.36}$$

and calling *torsion scalar* the following term

$$\hat{T} = \frac{1}{4}\big(\hat{T}^{\alpha\mu\nu}\hat{T}_{\alpha\mu\nu} + 2\hat{T}_{\alpha\mu\nu}\hat{T}^{\mu\alpha\nu}\big) - \hat{T}^\alpha\hat{T}_\alpha, \tag{4.3.37}$$

it follows that

$$\boxed{\overset{\circ}{R} = -\hat{T} - \tilde{B}}, \tag{4.3.38}$$

where

$$\tilde{B} = 2\overset{\circ}{\nabla}_\mu \hat{T}^\mu = \frac{2}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}\hat{T}^\mu\big), \tag{4.3.39}$$

is a boundary term.
We now recall the Einstein-Hilbert action of GR (see Sec.(2.2.1)),

$$S_{GR} = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}(\mathcal{L}_{EH} + \mathcal{L}_m) = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}(\overset{\circ}{R} + \mathcal{L}_m) \tag{4.3.40}$$

where $\mathcal{L}_{EH} = \overset{\circ}{R}(g)$ is the Einstein-Hilbert Lagrangian.
It is straightforward to see that if we assume a metric Teleparallel Gravity, we can transform the GR action into alternative formulations. Hence, up to a boundary term, which gives no contributions because the boundary is fixed and the variation there vanishes, a particular TG action assumes the form

$$S_{TEGR} = -\frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\,\mathcal{L}_{TG} + \int d^4x\sqrt{-g}\,\mathcal{L}_m = \frac{c^4}{16\pi G}\int d^4x\, e\hat{T} + \int d^4x\, e\mathcal{L}_m, \tag{4.3.41}$$

in which the TEGR Lagrangian takes the following form:

$$\mathcal{L}_{TEGR} = \frac{c^4 e}{16\pi G}\hat{T}. \tag{4.3.42}$$

The action (4.3.41) is *dynamically equivalent* to that of GR, in the sense that they lead to equivalent dynamics for the metric. At Lagrangian level:

$$\mathcal{L}_{TEGR} = \mathcal{L}_{GR} - \partial_\mu\left(\frac{ec^4}{8\pi G}\hat{T}^\mu\right). \tag{4.3.43}$$

This specific case of TG is called TEGR. We remark that this equivalence is possible only for a particular combination of coefficients of the action (4.3.41).

Let us analyze for a moment

$$\mathcal{L}_{TEGR} = \frac{c^4 e}{16\pi G}\left[\frac{1}{4}\left(\hat{T}^{\alpha\mu\nu}\hat{T}_{\alpha\mu\nu} + 2\hat{T}_{\alpha\mu\nu}\hat{T}^{\mu\alpha\nu}\right) - \hat{T}^{\alpha}\hat{T}_{\alpha}\right], \tag{4.3.44}$$

where we recall that $\hat{T}^{\rho}{}_{\mu\nu} = e_a{}^{\rho}\hat{T}^{a}{}_{\mu\nu}$ and the form of the torsion in TG (4.3.4)

$$\hat{T}^{a}{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \overset{\bullet}{\omega}{}^{a}{}_{b\mu} e^b{}_\nu - \overset{\bullet}{\omega}{}^{a}{}_{b\nu} e^b{}_\mu. \tag{4.3.45}$$

It has the usual form of any gauge theory Lagrangian, in which the first term is quadratic in the torsion, being the field strength of the theory. The remaining two terms are due to the fact that in TG we are dealing with the tetrad formalism, which allows us to have a relation between internal indeces and external indeces. For this reason, we can have additional contractions that otherwise cannot be allowed. Note also that the construction of scalar objects starting from the torsion is possible because it is a tensorial quantity, hence each term of the Lagrangian is invariant under both general coordinate and local Lorentz transformations.

### 4.3.4 TEGR Field Equations

The Teleparallel gravitational field equations are obtained performing a variation of the $\mathcal{L}_{TEGR}$ with respect to the tetrad field $e^a{}_\mu$, namely

$$\frac{\partial \mathcal{L}_{TEGR}}{\partial e^a{}_\mu} - \partial_\nu \frac{\partial \mathcal{L}_{TEGR}}{\partial(\partial_\nu e^a{}_\mu)} = 0. \tag{4.3.46}$$

The related field equations are[3]

$$\hat{G}_{\mu\nu} = \frac{1}{e}\partial_\lambda(e\hat{S}_{\mu\nu}{}^{\lambda}) - \frac{8\pi G}{c^4}\mathfrak{t}_{\mu\nu} = 0, \tag{4.3.47}$$

where $\hat{G}_{\mu\nu}$ is the TG Einstein tensor, while

$$\hat{S}_a{}^{\mu\nu} = -\hat{S}_a{}^{\nu\mu} = -\frac{8\pi G}{c^4 e}\frac{\partial \mathcal{L}_{TEGR}}{\partial(\partial_\nu e^a{}_\mu)} = \hat{K}^{\mu\nu}{}_a - e_a{}^{\nu}\hat{T}^{\mu} + e_a{}^{\mu}\hat{T}^{\nu} \tag{4.3.48}$$

[3]See Appendix A for a complete derivation.

is the *superpotential*, and

$$\mathfrak{t}_a{}^\mu = -\frac{1}{e}\frac{\partial \mathcal{L}_{TEGR}}{\partial e^a{}_\mu} = \frac{c^4}{8\pi G} e_a{}^\lambda \hat{S}_c{}^{\nu\mu}\hat{T}^c{}_{\nu\lambda} - \frac{1}{e} e_a{}^\mu \mathcal{L}_{TEGR} + \frac{c^4}{8\pi G}\overset{\bullet}{\omega}{}^c{}_{a\sigma}\hat{S}_c{}^{\mu\sigma} \tag{4.3.49}$$

is the *energy-momentum (pseudo) tensor* of the gravitational field[4]. Making use of eq.(4.3.47) and (4.3.49) it is possible to write the field equations in an alternative form:

$$\boxed{\hat{G}_{\mu\nu} = \frac{1}{e} e^a{}_\mu g_{\nu\rho}\partial_\sigma(e\hat{S}_a{}^{\rho\sigma}) - \hat{S}_b{}^\sigma{}_\nu \hat{T}^b{}_{\sigma\mu} + \frac{1}{2}\hat{T} g_{\mu\nu} - e^a{}_\mu \overset{\bullet}{\omega}{}^b{}_{a\sigma}\hat{S}_{b\nu}{}^\sigma = \frac{8\pi G}{c^4}\mathfrak{T}_{\mu\nu}}, \tag{4.3.50}$$

in which we have included the source matter term $\frac{8\pi G}{c^4}\mathfrak{T}_{\mu\nu}$.

The Noether theorem guarantees the existence of a conserved current for translations, which is obviously the energy-mometum pseudotensor $\mathfrak{t}_a{}^\mu$. Hence, it follows that

$$\partial_\mu \mathfrak{t}_a{}^\mu = 0. \tag{4.3.51}$$

Hence, taking the partial derivative $\partial_\mu$ on eq.(4.3.47) with a source term $\mathfrak{T}_{\mu\nu}$ and using the antisymmetry of the superpotential, we obtain

$$\partial_\mu \mathfrak{T}^{\mu\nu} = 0, \tag{4.3.52}$$

which shows that the energy-momentum tensor is conserved under ordinary derivative, which implies that the spacetime charges on hypersurfaces $x^0 = \text{const}$ $(\Sigma)$:

$$Q^\mu = \int_\Sigma d^3x \; e \; \mathfrak{T}^{0\mu} \tag{4.3.53}$$

are conserved.

#### 4.3.4.1 Equivalence with GR field equations

Now we will show that TEGR field equations are equivalent to those of GR. To do that, we will make use of the second Bianchi identity (1.7.11), namely

$$\nabla_\lambda R^\alpha{}_{\beta\mu\nu} + \nabla_\mu R^\alpha{}_{\beta\nu\lambda} + \nabla_\nu R^\alpha{}_{\beta\lambda\mu} = T^\rho{}_{\mu\lambda}R^\alpha{}_{\beta\nu\rho} + T^\rho{}_{\nu\lambda}R^\alpha{}_{\beta\mu\rho} + T^\rho{}_{\nu\mu}R^\alpha{}_{\beta\lambda\rho}. \tag{4.3.54}$$

[4]It is also possible to add a source term, namely $\hat{G}_{\mu\nu} = \frac{8\pi G}{c^4}\mathfrak{T}_{\mu\nu}$.

Since in TEGR curvature and non-metricity vanish and assuming the Weitzenböck gauge[5] , we can rewrite it as

$$\hat{\nabla}_\lambda R^\alpha{}_{\beta\mu\nu} + \hat{\nabla}_\mu R^\alpha{}_{\beta\nu\lambda} + \hat{\nabla}_\nu R^\alpha{}_{\beta\lambda\mu} = 0, \tag{4.3.55}$$

where we recall the TG Riemann tensor (4.3.31)

$$R^\alpha{}_{\beta\mu\nu} = \mathring{R}^\alpha{}_{\beta\mu\nu} + \underbrace{\mathring{\nabla}_\mu \hat{K}^\alpha{}_{\nu\beta} - \mathring{\nabla}_\nu \hat{K}^\alpha{}_{\mu\beta} + \hat{K}^\sigma{}_{\nu\beta}\hat{K}^\alpha{}_{\mu\sigma} - \hat{K}^\sigma{}_{\mu\beta}\hat{K}^\alpha{}_{\nu\sigma}}_{=\hat{\mathcal{K}}^\alpha{}_{\beta\mu\nu}} = \mathring{R}^\alpha{}_{\beta\mu\nu} + \hat{\mathcal{K}}^\alpha{}_{\beta\mu\nu} \tag{4.3.56}$$

where $\hat{\mathcal{K}}^\alpha{}_{\beta\mu\nu} = -\hat{\mathcal{K}}_\beta{}^\alpha{}_{\mu\nu}$ and $\hat{\mathcal{K}}^\alpha{}_{\beta\mu\nu} = -\hat{\mathcal{K}}^\alpha{}_{\beta\nu\mu}$. Thus, eq.(4.3.55) becomes

$$\hat{\nabla}_\lambda \mathring{R}^\alpha{}_{\beta\mu\nu} + \hat{\nabla}_\mu \mathring{R}^\alpha{}_{\beta\nu\lambda} + \hat{\nabla}_\nu \mathring{R}^\alpha{}_{\beta\lambda\mu} + \hat{\nabla}_\lambda \hat{\mathcal{K}}^\alpha{}_{\beta\mu\nu} + \hat{\nabla}_\mu \hat{\mathcal{K}}^\alpha{}_{\beta\nu\lambda} + \hat{\nabla}_\nu \hat{\mathcal{K}}^\alpha{}_{\beta\lambda\mu} = 0. \tag{4.3.57}$$

By contracting $\alpha$ and $\lambda$, raising the second index and eploting the properties of the Riemann tensor, it follows that

$$\hat{\nabla}_\mu (\mathring{R}^\mu{}_\nu + \hat{\mathcal{K}}^\mu{}_\nu) - \frac{1}{2}\hat{\nabla}_\nu (\mathring{R} + \hat{\mathcal{K}}) = 0 \tag{4.3.58}$$

where $\hat{\mathcal{K}}^\mu{}_\nu = \hat{\mathcal{K}}^\lambda{}_{\mu\lambda\nu}$ and $\hat{\mathcal{K}} = \hat{\mathcal{K}}^\nu{}_\nu$. Eqs.(4.3.58) implies that

$$\boxed{\mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R} = -\hat{\mathcal{K}}_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\hat{\mathcal{K}}}, \tag{4.3.59}$$

which tells us that TEGR field equations are equivalent to GR field equations and, imposing GR vacuum field equations,

$$\boxed{\hat{\mathcal{K}}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{\mathcal{K}} = 0}. \tag{4.3.60}$$

[5]We will see in Sec.(4.3.5) why this choice is always possible.

Eqs.(4.3.60) are the TEGR field equations and they can be shown to be equivalent to eqs.(4.3.50), hence

$$\hat{G}_{\mu\nu} = \frac{1}{e} e^{a}{}_{\mu} g_{\nu\rho} \partial_\sigma (e \hat{S}_a{}^{\rho\sigma}) - \hat{S}_b{}^{\sigma}{}_{\nu} \hat{T}^{b}{}_{\sigma\mu} + \frac{1}{2} \hat{T} g_{\mu\nu} - e^{a}{}_{\mu} \overset{\bullet}{\omega}{}^{b}{}_{a\sigma} \hat{S}_{b\nu}{}^{\sigma} = 0$$
$$\Updownarrow$$
$$\hat{\mathcal{K}}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{\mathcal{K}} = 0$$

In order to show this, let us first rewrite $\hat{\mathcal{K}}_{\mu\nu}$ and the scalar $\hat{\mathcal{K}}$.

$$\begin{aligned}\hat{\mathcal{K}}_{\mu\nu} = \hat{\mathcal{K}}^{\alpha}{}_{\mu\alpha\nu} &= \mathring{\nabla}_\alpha \hat{K}^{\alpha}{}_{\mu\nu} - \mathring{\nabla}_\nu \hat{K}^{\alpha}{}_{\mu\alpha} + \hat{K}^{\sigma}{}_{\mu\nu} \hat{K}^{\alpha}{}_{\sigma\alpha} - \hat{K}^{\sigma}{}_{\mu\alpha} \hat{K}^{\alpha}{}_{\sigma\nu} \\ &= \mathring{\nabla}_\alpha \hat{K}^{\alpha}{}_{\mu\nu} + \mathring{\nabla}_\nu \hat{T}_\mu - \hat{K}^{\sigma}{}_{\mu\nu} \hat{T}_\sigma - \hat{K}^{\sigma}{}_{\mu\alpha} \hat{K}^{\alpha}{}_{\sigma\nu} \\ &= \mathring{\nabla}_\alpha \hat{S}_\nu{}^{\alpha}{}_{\mu} + \mathring{\nabla}_\alpha \hat{T}^{\alpha} g_{\mu\nu} - \hat{K}^{\alpha}{}_{\sigma\nu} \hat{S}_\alpha{}^{\sigma}{}_{\mu},\end{aligned} \tag{4.3.61}$$

where we have used the expression of the contortion tensor (3.1.8) $\hat{K}^{\rho}{}_{\mu\nu} = \frac{1}{2}(\hat{T}_\mu{}^{\rho}{}_{\nu} + \hat{T}_\nu{}^{\rho}{}_{\mu} - \hat{T}^{\rho}{}_{\mu\nu})$ and the superpotential (4.3.48) $\hat{S}_a{}^{\mu\nu} = -\hat{S}_a{}^{\nu\mu} = \hat{K}^{\mu\nu}{}_a - e_a{}^{\nu} \hat{T}^{\mu} + e_a{}^{\mu} \hat{T}^{\nu}$ to retrieve the following components:

$$\begin{aligned}\hat{K}^{\alpha}{}_{\mu\alpha} &= \frac{1}{2}\big(\overbrace{\hat{T}_\mu{}^{\alpha}{}_{\alpha}}^{=0} + \overbrace{\hat{T}_\alpha{}^{\alpha}{}_{\mu}}^{-\hat{T}^{\alpha}{}_{\mu\alpha}} - \hat{T}^{\alpha}{}_{\mu\alpha}\big) = -\hat{T}^{\alpha}{}_{\mu\alpha} = -\hat{T}_\mu, \\ \hat{K}^{\alpha}{}_{\alpha\mu} &= \frac{1}{2}\big(\hat{T}_\alpha{}^{\alpha}{}_{\mu} + \hat{T}_\mu{}^{\alpha}{}_{\alpha} - \hat{T}^{\alpha}{}_{\alpha\mu}\big) = 0, \\ \hat{K}^{\mu\nu}{}_a &= \hat{S}_a{}^{\mu\nu} + e_a{}^{\nu} \hat{T}^{\mu} - e_a{}^{\mu} \hat{T}^{\nu} \implies \hat{K}^{\mu\nu}{}_\lambda = \hat{S}_\lambda{}^{\mu\nu} + \delta^{\nu}{}_{\lambda} \hat{T}^{\mu} - \delta^{\mu}{}_{\lambda} \hat{T}^{\nu}.\end{aligned}$$

Instead, the scalar $\mathcal{K}$ assumes the form

$$\begin{aligned}\hat{\mathcal{K}} = g^{\mu\nu} \hat{\mathcal{K}}_{\mu\nu} &= \mathring{\nabla}_\alpha \hat{S}^{\mu\alpha}{}_{\mu} + \mathring{\nabla}_\alpha \hat{T}^{\alpha} g_{\mu\nu} g^{\mu\nu} - \hat{K}^{\alpha}{}_{\sigma}{}^{\mu} \hat{S}_\alpha{}^{\sigma}{}_{\mu} \\ &= 2\mathring{\nabla}_\alpha \hat{T}^{\alpha} + \hat{T}.\end{aligned} \tag{4.3.62}$$

We can now insert (4.3.61) and (4.3.62) into (4.3.60):

$$\begin{aligned}&\mathring{\nabla}_\alpha \hat{S}_\nu{}^{\alpha}{}_{\mu} + \mathring{\nabla}_\alpha \hat{T}^{\alpha} g_{\mu\nu} - \hat{K}^{\alpha}{}_{\sigma\nu} \hat{S}_\alpha{}^{\sigma}{}_{\mu} - \frac{1}{2} g_{\mu\nu} \big(2\mathring{\nabla}_\alpha \hat{T}^{\alpha} + \hat{T}\big) = 0 \\ &\implies \mathring{\nabla}_\alpha \hat{S}_{\nu\mu}{}^{\alpha} + \hat{K}^{\alpha}{}_{\sigma\nu} \hat{S}_\alpha{}^{\sigma}{}_{\mu} + \frac{1}{2} g_{\mu\nu} \hat{T} = 0.\end{aligned} \tag{4.3.63}$$

We can retrive the same equation (4.3.63) starting from eq.(4.3.50), namely

$$\frac{1}{e}e^a{}_\mu g_{\nu\rho}\partial_\sigma(e\hat{S}_a{}^{\rho\sigma}) - \hat{S}_b{}^\sigma{}_\nu \hat{T}^b{}_{\sigma\mu} + \frac{1}{2}\hat{T}g_{\mu\nu} - e^a{}_\mu \overset{\bullet}{\omega}{}^b{}_{a\sigma}\hat{S}_{b\nu}{}^\sigma = 0. \tag{4.3.64}$$

First of all, applying the Leibniz rule and exploiting the identity

$$\frac{1}{e}\partial_\sigma e = \partial_\sigma \ln\sqrt{-g} = \mathring{\Gamma}^\alpha{}_{\alpha\sigma}, \tag{4.3.65}$$

we can rewrite the first term as

$$\begin{aligned}
\frac{1}{e}e^a{}_\mu g_{\nu\rho}\partial_\sigma(e\hat{S}_a{}^{\rho\sigma}) &= e^a{}_\mu g_{\nu\rho}\partial_\sigma \hat{S}_a{}^{\rho\sigma} + \frac{1}{e}(\partial_\sigma e)e^a{}_\mu g_{\nu\rho}\hat{S}_a{}^{\rho\sigma} \\
&= e^a{}_\mu g_{\nu\rho}\left[\hat{S}_\alpha{}^{\rho\sigma}\partial_\sigma e_a{}^\alpha + e_a{}^\alpha \partial_\sigma S_\alpha{}^{\rho\sigma}\right] + \mathring{\Gamma}^\alpha{}_{\alpha\sigma}e^a{}_\mu g_{\nu\rho}\hat{S}_a{}^{\rho\sigma} \\
&= e^a{}_\mu\big(\partial_\sigma e_a{}^\alpha\big)\hat{S}_{\alpha\nu}{}^\sigma + g_{\nu\rho}\partial_\sigma \hat{S}_\mu{}^{\rho\sigma} + \mathring{\Gamma}^\alpha{}_{\alpha\sigma}\hat{S}_{\mu\nu}{}^\sigma \\
&= e^a{}_\mu\left[\overset{\bullet}{\omega}{}^b{}_{a\sigma}e_b{}^\alpha - \mathring{\Gamma}^\alpha{}_{\sigma\lambda}e_a{}^\lambda\right]\hat{S}_{\alpha\nu}{}^\sigma + \partial_\sigma \hat{S}_{\mu\nu}{}^\sigma - \hat{S}_\mu{}^{\rho\sigma}\partial_\sigma g_{\nu\rho} + \mathring{\Gamma}^\alpha{}_{\alpha\sigma}\hat{S}_{\mu\nu}{}^\sigma \\
&= e^a{}_\mu\overset{\bullet}{\omega}{}^b{}_{a\sigma}e_b{}^\alpha\hat{S}_{\alpha\nu}{}^\sigma - \mathring{\Gamma}^\alpha{}_{\sigma\mu}\hat{S}_{\alpha\nu}{}^\sigma + \partial_\sigma\hat{S}_{\mu\nu}{}^\sigma \\
&\qquad - \hat{S}_\mu{}^{\rho\sigma}\left[\Gamma^\lambda{}_{\sigma\nu}g_{\lambda\rho} + \Gamma^\lambda{}_{\sigma\rho}g_{\nu\lambda}\right] + \mathring{\Gamma}^\alpha{}_{\alpha\sigma}\hat{S}_{\mu\nu}{}^\sigma,
\end{aligned} \tag{4.3.66}$$

where in the fourth line we have used the tetrad postulate (1.14.7) to rewrite the partial derivative of the tetrad and the compatibility of the metric (1.14.8).
Inserting (4.3.66) into (4.3.64), it follows

$$\begin{aligned}
&\cancel{e^a{}_\mu\overset{\bullet}{\omega}{}^b{}_{a\sigma}e_b{}^\alpha\hat{S}_{\alpha\nu}{}^\sigma} - \hat{\Gamma}^\alpha{}_{\sigma\mu}\hat{S}_{\alpha\nu}{}^\sigma + \partial_\sigma\hat{S}_{\mu\nu}{}^\sigma - \hat{S}_\mu{}^{\rho\sigma}\left[\hat{\Gamma}^\lambda{}_{\sigma\nu}g_{\lambda\rho} + \hat{\Gamma}^\lambda{}_{\sigma\rho}g_{\nu\lambda}\right] \\
&\qquad\qquad + \mathring{\Gamma}^\alpha{}_{\alpha\sigma}\hat{S}_{\mu\nu}{}^\sigma - \hat{S}_b{}^\sigma{}_\nu\hat{T}^b{}_{\sigma\mu} + \frac{1}{2}\hat{T}g_{\mu\nu} - \cancel{e^a{}_\mu\overset{\bullet}{\omega}{}^b{}_{a\sigma}\hat{S}_{b\nu}{}^\sigma} = 0, \\
&\partial_\sigma\hat{S}_{\mu\nu}{}^\sigma - \hat{\Gamma}^\alpha{}_{\sigma\mu}\hat{S}_{\alpha\nu}{}^\sigma - \hat{\Gamma}^\lambda{}_{\sigma\nu}\hat{S}_{\mu\lambda}{}^\sigma - \hat{\Gamma}^\lambda{}_{\sigma\rho}\hat{S}_\mu{}^{\rho\sigma}g_{\nu\lambda} - \hat{S}_b{}^\sigma{}_\nu\hat{T}^b{}_{\sigma\mu} + \mathring{\Gamma}^\alpha{}_{\alpha\sigma}\hat{S}_{\mu\nu}{}^\sigma + \frac{1}{2}\hat{T}g_{\mu\nu} = 0.
\end{aligned}$$

At this point, we can insert the decomposition of the STG connection, i.e. $\hat{\Gamma}^\alpha{}_{\mu\nu} = \mathring{\Gamma}^\alpha{}_{\mu\nu} + \hat{K}^\alpha{}_{\mu\nu}$ and assemble the Levi-Civita covariant derivative:

$$\begin{aligned}
&\underline{\partial_\sigma\hat{S}_{\mu\nu}{}^\sigma} - \underline{\mathring{\Gamma}^\alpha{}_{\sigma\mu}\hat{S}_{\alpha\nu}{}^\sigma} - \hat{K}^\alpha{}_{\sigma\mu}\hat{S}_{\alpha\nu}{}^\sigma - \underline{\mathring{\Gamma}^\lambda{}_{\sigma\nu}\hat{S}_{\mu\lambda}{}^\sigma} - \hat{K}^\lambda{}_{\sigma\nu}\hat{S}_{\mu\lambda}{}^\sigma - \mathring{\Gamma}^\lambda{}_{\sigma\rho}\hat{S}_\mu{}^{\rho\sigma}g_{\nu\lambda} \\
&\qquad\qquad - \hat{K}^\lambda{}_{\sigma\rho}\hat{S}_\mu{}^{\rho\sigma}g_{\nu\lambda} - \hat{S}_\alpha{}^\sigma{}_\nu\hat{T}^\alpha{}_{\sigma\mu} + \underline{\mathring{\Gamma}^\alpha{}_{\alpha\sigma}\hat{S}_{\mu\nu}{}^\sigma} + \frac{1}{2}\hat{T}g_{\mu\nu} = 0.
\end{aligned} \tag{4.3.67}$$

By the antisymmetry of $\hat{S}_\alpha{}^{\mu\nu}$ and $\hat{K}^\alpha{}_{\mu\nu}$, we have a further semplification of some terms, namely

$$\begin{aligned}
&\mathring{\Gamma}^\lambda{}_{\sigma\rho}\hat{S}_\mu{}^{\rho\sigma}g_{\nu\lambda} = 0,\\
&\hat{K}_{\nu\sigma\rho}\hat{S}_\mu{}^{\rho\sigma} + \hat{K}^\lambda{}_{\sigma\nu}\hat{S}_{\mu\lambda}{}^\sigma = 0,\\
&\hat{K}^\alpha{}_{\sigma\mu}\hat{S}_\alpha{}^\sigma{}_\nu - \hat{S}_\alpha{}^\sigma{}_\nu\hat{T}^\alpha{}_{\sigma\mu} = \frac{1}{2}\left[\hat{T}_\sigma{}^\alpha{}_\mu + \hat{T}_\mu{}^\alpha{}_\sigma - \hat{T}^\alpha{}_{\sigma\mu}\right]\hat{S}_\alpha{}^\sigma{}_\nu - \hat{S}_\alpha{}^\sigma{}_\nu\hat{T}^\alpha{}_{\sigma\mu} = \hat{K}^\alpha{}_{\sigma\mu}\hat{S}_\alpha{}^\sigma{}_\nu.
\end{aligned}$$

Then, from eq.(4.3.67) we obtain

$$\mathring{\nabla}_\alpha\hat{S}_{\mu\nu}{}^\alpha + \hat{K}^\alpha{}_{\sigma\mu}\hat{S}_\alpha{}^\sigma{}_\nu + \frac{1}{2}g_{\mu\nu}\hat{T} = 0,$$

hence we have retrieved eq.(4.3.63) and shown the equivalence of the field equations.

### 4.3.5 The role of the spin connection in the TEGR dynamics

Since the GR Levi-Civita connection is a function of the tetrad, the Einstein-Hilbert action can be expressed in terms of tetrad only [25]. Therefore, the same Einstein-Hilbert action corresponds to both $S_{TEGR}(e^a{}_\mu, 0)$ and $S_{TEGR}(e^a{}_\mu, \overset{\bullet}{\omega}{}^a{}_{a\mu})$, written with the same tetrad. Hence, the following identity holds (see eq.(4.3.43)):

$$\mathcal{L}_{TEGR}(e^a{}_\mu, \overset{\bullet}{\omega}{}^a{}_{b\mu}) + \partial_\mu\left(\frac{ec^4}{8\pi G}\hat{T}^\mu(e^a{}_\mu, \overset{\bullet}{\omega}{}^a{}_{b\mu})\right) = \mathcal{L}_{TEGR}(e^a{}_\mu, 0) + \partial_\mu\left(\frac{ec^4}{8\pi G}\hat{T}^\mu(e^a{}_\mu, 0)\right). \tag{4.3.68}$$

If we contract the TG torsion tensor (4.3.4) with $e_a{}^\nu$, we obtain

$$\begin{aligned}
\hat{T}_\mu(e^a{}_\mu, \overset{\bullet}{\omega}{}^a{}_{b\mu}) &= e_a{}^\nu\hat{T}^a{}_{\mu\nu}(e^a{}_\mu, \overset{\bullet}{\omega}{}^a{}_{b\mu})\\
&= e_a{}^\nu\partial_\mu e^a{}_\nu - e_a{}^\nu\partial_\nu e^a{}_\mu + \underbrace{e_a{}^\nu\overset{\bullet}{\omega}{}^a{}_{b\mu}e^b{}_\nu}_{=0} - e_a{}^\nu\overset{\bullet}{\omega}{}^a{}_{b\nu}e^b{}_\mu = \hat{T}_\mu(e^a{}_\mu, 0) - \overset{\bullet}{\omega}_\mu,
\end{aligned} \tag{4.3.69}$$

where $\overset{\bullet}{\omega}_\mu = e_a{}^\nu\overset{\bullet}{\omega}{}^a{}_{b\nu}e^b{}_\mu$, and the null term is so due to the contraction of antisymmetric indeces. Combining (4.3.68) with (4.3.69) we find that the Lagrangians are related by a total divergence only, namely

$$\mathcal{L}_{TEGR}(e^a{}_\mu, \overset{\bullet}{\omega}{}^a{}_{b\mu}) = \mathcal{L}_{TEGR}(e^a{}_\mu, 0) + \partial_\mu\left(\frac{ec^4}{8\pi G}\overset{\bullet}{\omega}{}^\mu\right). \tag{4.3.70}$$

This relation shows the spin connection $\overset{\bullet}{\omega}{}^a{}_{b\mu}$ enters the Lagrangian as a total derivative, thus the variation with respect to it is identically null

$$\frac{\delta \mathcal{L}_{TEGR}}{\delta \overset{\bullet}{\omega}{}^a{}_{b\mu}} = 0, \tag{4.3.71}$$

meaning that the field equations obtained from $\mathcal{L}_{TEGR}(e^a{}_\mu, \overset{\bullet}{\omega}{}^a{}_{b\mu})$ or $\mathcal{L}_{TEGR}(e^a{}_\mu, 0)$ are the same and can be solved independently of the spin connection. Hence, the spin connection does not contribute to the field equations and this justifies the assumption of the Weitzenböck gauge to reduce the calculations.

A direct consequence is that the TEGR field equations determine only the equivalence class of tetrads with respect to the local Lorentz transformations $\Lambda^a{}_b(x)$. Thus, the field equations do not determine $\Lambda^a{}_b(x)$, which means that we are able to determine the tetrads up to a local Lorentz transformation and thus to determine only the metric tensor.

Nevertheless, the spin connection plays a fundamental role in TEGR, because it garantees the covariance of the action under local Lorentz transformations and diffeormorphisms. In fact, in the count of the degrees of freedom (DoFs), we start from the 16 of the tetrad $e^a{}_\mu$, from which we have to subtract 6 DoFs related to inertial effects due to the spin connection and 8 non-dynamical DoFs due to diffeomorphisms (as in GR[6]). At the end, TEGR remains with 2 DoFs as in the case of GR, and this is a further motivation for the dynamics equivalence between TEGR and GR.

[6]In GR we have the 10 DoFs of the metric tensor components from which we subtract the 8 DoFs of invariance under diffeomorphisms.

# Chapter 5

# The Symmetric Teleparallel Gravity

In this chapter, we will deal with the last member of the Trinity of Gravity, which is the *Symmetric Teleparallel Gravity (STG)*. This theory presents the transition from a curvature-based description gravity to one based on non-metricity, in fact STG is characterized by having a curvature and a torsion tensor identically null, i.e. $R^{\rho}{}_{\sigma\mu\nu} = T^{\rho}{}_{\mu\nu} = 0$. As we saw, we can define a non-metricity tensor (1.7.6), which represents the gravitational effects

$$\overset{\diamond}{Q}_{\lambda\mu\nu} = \overset{\diamond}{\nabla}_{\lambda} g_{\mu\nu}, \tag{5.0.1}$$

or equivalently

$$\overset{\diamond}{Q}_{\lambda}{}^{\mu\nu} = -\overset{\diamond}{\nabla}_{\lambda} g^{\mu\nu}, \tag{5.0.2}$$

$$\overset{\diamond}{Q}{}^{\lambda\mu\nu} = -g^{\lambda\rho}\overset{\diamond}{\nabla}_{\rho} g^{\mu\nu}. \tag{5.0.3}$$

The main difference with TG is that STG provides the metric as the fundamental dynamics object.
Before delving into the theory, it is important to recall some geometric effects and implications due to the presence of non-metricity:

- Raising up or lowering down indeces of tensorial objects under the covariant derivative $\overset{\diamond}{\nabla}$ produces now an additional term due to the non-metrical compatibility, hence for a vector $v^{\mu}$,

$$\boxed{g_{\nu\lambda}\overset{\diamond}{\nabla}_{\mu} v^{\lambda} = \overset{\diamond}{\nabla}_{\mu} v_{\nu} - v^{\lambda}\overset{\diamond}{Q}_{\mu\nu\lambda}}. \tag{5.0.4}$$

- As it could be seen in Fig.(3.1), non-metricity does not preserve the length of a vector. Let us consider two vectors $v = v^{\mu}\partial_{\mu}$ and $w = w^{\mu}\partial_{\mu}$ parallel along a curve $\gamma$, with tangent vectors $T = T^{\mu}\partial_{\mu}$, where $T^{\mu} = \dot{\gamma}^{\mu}$. Hence it means that $\overset{\diamond}{\nabla}_{T} v^{\mu} = T^{\lambda}\overset{\diamond}{\nabla}_{\lambda} v^{\mu} = 0$

and $T^\lambda \overset{\diamond}{\nabla}_\lambda w^\mu = 0$. The scalar product of the two vectors, namely $v \cdot w = g_{\mu\nu} v^\mu w^\nu$ is not conserved after a parallel transport of the two vectors along the curve $\gamma$. In fact:

$$\begin{aligned} T^\lambda \overset{\diamond}{\nabla}_\lambda (v \cdot w) &= T^\lambda \overset{\diamond}{\nabla}_\lambda (g_{\mu\nu} v^\mu w^\nu) \\ &= T^\lambda (\overset{\diamond}{\nabla}_\lambda g_{\mu\nu}) v^\mu w^\nu + \overbrace{T^\lambda (\overset{\diamond}{\nabla}_\lambda v^\mu)}^{=0} g_{\mu\nu} w^\nu + \overbrace{T^\lambda (\overset{\diamond}{\nabla}_\lambda v^\nu)}^{=0} g_{\mu\nu} w^\mu \\ &= T^\lambda v^\mu w^\nu \overset{\diamond}{Q}_{\lambda\mu\nu}. \end{aligned} \tag{5.0.5}$$

Furthermore, if $v = w$, if follows that

$$T^\lambda \overset{\diamond}{\nabla}_\lambda (g_{\mu\nu} v^\mu v^\nu) = T^\lambda v^\mu v^\nu \overset{\diamond}{Q}_{\lambda\mu\nu}, \tag{5.0.6}$$

hence, the norm $|v| = \sqrt{v \cdot v}$ is conserved, making it not normalizable.
An important consequence is that is not possible to define a proper time along a curve, because the notion of distance and duration depends on the metric structure that is no longer preserved by the connection used in STG.

- Besides the acceleration $a^\mu$ of a four-vector $v^\mu$, it is possible to define the *anomalous acceleration* $\tilde{a}_\mu$:

$$a^\mu = u^\lambda \overset{\diamond}{\nabla}_\lambda u^\mu, \tag{5.0.7}$$

$$\tilde{a}_\mu = u^\lambda \overset{\diamond}{\nabla}_\lambda u_\mu = u^\lambda (g_{\mu\alpha} \overset{\diamond}{\nabla}_\lambda u^\alpha + u^\alpha \overset{\diamond}{Q}_{\lambda\mu\alpha}) = a_\mu + \overset{\diamond}{Q}_{\lambda\mu\alpha} u^\lambda u^\alpha. \tag{5.0.8}$$

Remarkably is the fact that, under these conditions, the four-velocity is not orthogonal to the four-acceleration anymore, in fact:

$$\begin{aligned} u_\mu a^\mu &= u_\mu u^\lambda \overset{\diamond}{\nabla}_\lambda u^\mu \\ &= u^\lambda \overset{\diamond}{\nabla}_\lambda (u_\mu u^\mu) - u^\mu u^\lambda \overset{\diamond}{\nabla}_\lambda u_\mu \\ &= u^\lambda \overset{\diamond}{\nabla}_\lambda (g_{\mu\nu} u^\mu u^\nu) - u^\mu \tilde{a}_\mu \\ &= u^\lambda (\overset{\diamond}{\nabla}_\lambda g_{\mu\nu}) u^\mu u^\nu + u^\lambda (\overset{\diamond}{\nabla}_\lambda v^\mu) g_{\mu\nu} u^\nu + u^\lambda (\overset{\diamond}{\nabla}_\lambda v^\nu) g_{\mu\nu} u^\mu - u^\mu \tilde{a}_\mu \\ &= u^\lambda \overset{\diamond}{Q}_{\lambda\mu\nu} u^\mu u^\nu + a^\nu g_{\mu\nu} u^\mu + a^\mu g_{\mu\nu} u^\nu - u^\mu \tilde{a}_\mu \\ &= \overset{\diamond}{Q}_{\lambda\mu\nu} u^\lambda u^\mu u^\nu + 2 u_\mu a^\mu - u^\mu \tilde{a}_\mu. \end{aligned} \tag{5.0.9}$$

It follows that

$$(\tilde{a}_\mu - a_\mu)u^\mu = \overset{\diamond}{Q}_{\lambda\mu\nu}u^\lambda u^\mu u^\nu, \tag{5.0.10}$$

hence the non-metricity tensor quantifies how much the anomalous acceleration deviates from the usual one.

- If the curve $\gamma$ is an STG autoparallel, now we have that it has an acceleration: the anomalous one, namely

$$a^\mu = u^\lambda \overset{\diamond}{\nabla}_\lambda u^\mu = 0 \implies \tilde{a}_\mu = u^\lambda \overset{\diamond}{\nabla}_\lambda u_\mu = \overset{\diamond}{Q}_{\lambda\nu\mu}u^\lambda u^\nu. \tag{5.0.11}$$

Note that in order to recover the length conservation (5.0.6) and the autoparallel definition (3.4.1) imposing the following constraints:

$$\overset{\diamond}{Q}_{(\lambda\mu\nu)} = 0 \implies u^\lambda \overset{\diamond}{\nabla}_\lambda (g_{\mu\nu}u^\mu u^\nu) = \overset{\diamond}{Q}_{(\lambda\mu\nu)}u^\lambda u^\mu u^\nu = 0 \overset{(5.0.10)}{\implies} \tilde{a}_\mu u^\mu = a_\mu u^\mu, \tag{5.0.12}$$

$$\overset{\diamond}{Q}_{(\lambda\mu)\nu} = 0 \implies \tilde{a}_\nu = \overset{\diamond}{Q}_{(\lambda\mu)\nu}u^\lambda u^\mu = 0. \tag{5.0.13}$$

These two constraints are too strict to be imposed, in fact this issue is solved by resorting to the *Weyl conformal transformations* (see Ref. [26] for details).

## 5.1 Symmetric Teleparallel Equivalent of General Relativity (STEGR)

Also in this case, the generalized affine connection is flat but $K^\alpha{}_{\mu\nu} = 0$ this time. The Symmetric Teleparallel Gravity (STG) affine connection is then

$$\boxed{\overset{\diamond}{\Gamma}{}^\alpha{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} + \overset{\diamond}{L}{}^\alpha{}_{\mu\nu}}. \tag{5.1.1}$$

The Riemann tensor (4.3.28) is now

$$R^\alpha{}_{\beta\mu\nu} = \overset{\circ}{R}{}^\alpha{}_{\beta\mu\nu} + \overset{\circ}{\nabla}_\mu \overset{\diamond}{L}{}^\alpha{}_{\nu\beta} - \overset{\circ}{\nabla}_\nu \overset{\diamond}{L}{}^\alpha{}_{\mu\beta} + \overset{\diamond}{L}{}^\alpha{}_{\mu\sigma}\overset{\diamond}{L}{}^\sigma{}_{\nu\beta} - \overset{\diamond}{L}{}^\alpha{}_{\nu\sigma}\overset{\diamond}{L}{}^\sigma{}_{\mu\beta} = 0, \tag{5.1.2}$$

from which we can find the Ricci scalar by contracting $\alpha$ with $\mu$ and raising up and contracting the remaining indeces:

$$\overset{\circ}{R} + \overset{\circ}{\nabla}_\mu\big(\overset{\diamond}{L}^\mu - \overset{\diamond}{\tilde{L}}^\mu\big) + \overset{\diamond}{\tilde{L}}_\mu \overset{\diamond}{L}^\mu - \overset{\diamond}{L}_{\mu\nu\sigma}\overset{\diamond}{L}^{\sigma\mu\nu} = 0, \tag{5.1.3}$$

where we named, using eq.(3.1.9) $L^\rho{}_{\mu\nu} := \frac{1}{2}(Q^\rho{}_{\mu\nu} - Q_\mu{}^\rho{}_\nu - Q_\nu{}^\rho{}_\mu)$,

$$\overset{\diamond}{L}^\mu = \overset{\diamond}{L}^\mu{}_\nu{}^\nu = \frac{1}{2}\big(\overset{\diamond}{Q}^\mu{}_{\nu\nu} - \overset{\diamond}{Q}_\nu{}^\mu{}_\nu - \overset{\diamond}{Q}_\nu{}^\mu{}_\nu\big) = \frac{1}{2}\big(\overset{\diamond}{Q}^\mu - 2\overset{\diamond}{\tilde{Q}}^\mu\big), \tag{5.1.4}$$

$$\overset{\diamond}{\tilde{L}}^\mu = \overset{\diamond}{L}^{\nu\mu}{}_\nu = \overset{\diamond}{L}^\nu{}_\nu{}^\mu = \frac{1}{2}\big(\overset{\diamond}{Q}^{\mu\nu}{}_\nu - \overset{\diamond}{Q}^{\mu\nu}{}_\nu - \overset{\diamond}{Q}_\nu{}^\nu{}_\mu\big) = -\frac{1}{2}\overset{\diamond}{Q}^\mu, \tag{5.1.5}$$

where $\overset{\diamond}{Q}_\nu{}^{\mu\nu} = \overset{\diamond}{\tilde{Q}}^\mu$ and $\overset{\diamond}{Q}^\mu{}_\nu{}^\nu = \overset{\diamond}{Q}^\mu$. We can manipulate the last three terms of eq.(5.1.3):

$$\begin{aligned}\overset{\circ}{\nabla}_\mu\big(\overset{\diamond}{L}^\mu - \overset{\diamond}{\tilde{L}}^\mu\big) &= \overset{\circ}{\nabla}_\mu\big(\frac{1}{2}\overset{\diamond}{Q}^\mu - \overset{\diamond}{\tilde{Q}}^\mu - \frac{1}{2}\overset{\diamond}{Q}^\mu\big)\\ &= \overset{\circ}{\nabla}_\mu\big(\overset{\diamond}{Q}^\mu - \overset{\diamond}{\tilde{Q}}^\mu\big)\\ &= \frac{1}{\sqrt{-g}}\partial_\mu\big[\sqrt{-g}(\overset{\diamond}{Q}^\mu - \overset{\diamond}{\tilde{Q}}^\mu)\big] = B,\end{aligned} \tag{5.1.6}$$

that is a boundary term. Then,

$$\begin{aligned}\overset{\diamond}{\tilde{L}}_\mu \overset{\diamond}{L}^\mu - \overset{\diamond}{L}_{\mu\nu\sigma}\overset{\diamond}{L}^{\sigma\mu\nu} &= \frac{1}{4}\overset{\diamond}{Q}_\mu\big(\overset{\diamond}{Q}^\mu - 2\overset{\diamond}{\tilde{Q}}^\mu\big) - \frac{1}{4}\big(\overset{\diamond}{Q}_{\mu\nu\sigma} - \overset{\diamond}{Q}_{\nu\mu\sigma} - \overset{\diamond}{Q}_{\sigma\mu\nu}\big)\big(\overset{\diamond}{Q}^{\sigma\mu\nu} - \overset{\diamond}{Q}^{\mu\sigma\nu} - \overset{\diamond}{Q}^{\nu\sigma\mu}\big)\\ &= \frac{1}{4}\overset{\diamond}{Q}_\mu\overset{\diamond}{Q}^\mu - \frac{1}{2}\overset{\diamond}{Q}_\mu\overset{\diamond}{\tilde{Q}}^\mu + \frac{1}{2}\overset{\diamond}{Q}_{\mu\nu\sigma}\overset{\diamond}{Q}^{\nu\mu\sigma} - \frac{1}{4}\overset{\diamond}{Q}_{\mu\nu\sigma}\overset{\diamond}{Q}^{\mu\nu\sigma} = \overset{\diamond}{Q}\end{aligned} \tag{5.1.7}$$

which we addressed the name of *non-metricity scalar*. Finally, using eqs.(5.1.6) and (5.3.13), it is possible to rewrite the curvature tensor (5.1.3):

$$\boxed{\overset{\circ}{R} = \overset{\diamond}{Q} - B}. \tag{5.1.8}$$

As in the TEGR case, eq.(5.1.8) leads to the dynamical equivalence between GR and a specific STG Lagrangian, made up with the non-metricity scalar. In fact, the GR action leads to the same field equations as the ones derived from

$$S_{STEGR} = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\mathring{R} + \int d^4x\sqrt{-g}\mathcal{L}_m = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\overset{\diamond}{Q} + \int d^4x\sqrt{-g}\mathcal{L}_m, \tag{5.1.9}$$

where we have neglected the boundary term $B$ and the STEGR Lagrangian assumes the form

$$\mathcal{L}_{STEGR} = \frac{c^4\sqrt{-g}}{16\pi G}\overset{\diamond}{Q}. \tag{5.1.10}$$

Thus, at Lagrangian level we can write

$$\mathcal{L}_{STEGR} = \mathcal{L}_{GR} + \partial_\mu\left(\frac{\sqrt{-g}c^4}{16\pi G}(\overset{\diamond}{Q}^\mu - \overset{\diamond}{\tilde{Q}}^\mu)\right). \tag{5.1.11}$$

## 5.2 The coincident gauge

Before delving into the computation of the STEGR field equations, we want to observe a particular gauge that allows to simplify the calculations of the following sections.

The generalized affine connection coefficients (3.1.7) transform under a general coordinate map $\xi^\lambda \longrightarrow x^\lambda$ as

$$\Gamma^\rho{}_{\mu\nu}(x^\lambda) = \frac{\partial x^\rho}{\partial \xi^\gamma}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\Gamma^\gamma{}_{\alpha\beta}(\xi^\lambda) + \frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{\partial x^\rho}{\partial \xi^\alpha}. \tag{5.2.1}$$

The transformed affine connection

$$\Gamma^\alpha{}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \xi^\lambda}\partial_\mu\partial_\nu\xi^\lambda \tag{5.2.2}$$

is still flat, and this restricts the connection to be purely inertial, allowing the following form for eq.(5.2.2):

$$\Gamma^\alpha{}_{\mu\nu} = \Lambda_\lambda{}^\alpha\partial_\mu\Lambda^\lambda{}_\nu \tag{5.2.3}$$

where $\Lambda \in GL(4,\mathbb{R})$. We can then rewrite the torsion tensor as

$$T^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\mu} = \Lambda_\lambda{}^\alpha\partial_\mu\Lambda^\lambda{}_\nu - \Lambda_\lambda{}^\alpha\partial_\nu\Lambda^\lambda{}_\mu, \tag{5.2.4}$$

and if we impose the torsion-less condition of STEGR, we obtain

$$\partial_\mu \Lambda^\lambda{}_\nu - \partial_\nu \Lambda^\lambda{}_\mu = 0. \tag{5.2.5}$$

This expression is satisfied by introducing functions of the $x$ coordinates, i.e. $\xi^\mu = \xi^\mu(x^\nu)$ and defining

$$\Lambda^\mu{}_\nu = \partial_\nu \xi^\mu. \tag{5.2.6}$$

In this way, we obtain the flat and torsion-less connection of STG:

$$\overset{\diamond}{\Gamma}{}^\alpha{}_{\mu\nu} = \Lambda_\lambda{}^\alpha \partial_\mu \Lambda^\lambda{}_\nu = \frac{\partial \xi^\alpha}{\partial x^\lambda} \partial_\mu \frac{\partial \xi^\lambda}{\partial x^\nu}, \tag{5.2.7}$$

and it is clear that it can be set to zero under diffemorphisms: this connection is a pure-gauge connection, hence it is always possible to pick a coordinate system in which the connection vanishes.
For example, a null connection can be obtained by setting $\xi^\mu = x^\mu$. This is called *coincident gauge*.

The coincident gauge allows to further simplify the final form of the STG affine connection: we already know from (5.1.1) that

$$\overset{\diamond}{\Gamma}{}^\alpha{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} + \overset{\diamond}{L}{}^\alpha{}_{\mu\nu},$$

and if we now impose the coincident gauge, i.e.

$$\overset{\diamond}{\Gamma}{}^\alpha{}_{\mu\nu} = 0, \tag{5.2.8}$$

we obtain

$$\overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} = -\overset{\diamond}{L}{}^\alpha{}_{\mu\nu}. \tag{5.2.9}$$

A straightforward consequence is that the LC covariant derivative of a tensor $X^\mu{}_\nu$ is now

$$\begin{aligned} \overset{\circ}{\nabla}_\alpha X^\mu{}_\nu &= \partial_\alpha X^\mu{}_\nu + \overset{\circ}{\Gamma}{}^\mu{}_{\alpha\lambda} X^\lambda{}_\nu - \overset{\circ}{\Gamma}{}^\lambda{}_{\alpha\nu} X^\mu{}_\lambda \\ &= \partial_\alpha X^\mu{}_\nu - \overset{\diamond}{L}{}^\mu{}_{\alpha\lambda} X^\lambda{}_\nu + \overset{\diamond}{L}{}^\lambda{}_{\alpha\nu} X^\mu{}_\lambda, \end{aligned} \tag{5.2.10}$$

while the STG covariant derivative

$$\begin{aligned}\overset{\diamond}{\nabla}_\alpha X^\mu{}_\nu &= \partial_\alpha X^\mu{}_\nu + \overset{\diamond}{\Gamma}{}^\mu{}_{\alpha\lambda} X^\lambda{}_\nu - \overset{\diamond}{\Gamma}{}^\lambda{}_{\alpha\nu} X^\mu{}_\lambda \\ &= \partial_\alpha X^\mu{}_\nu. \end{aligned} \tag{5.2.11}$$

Let us make some considerations. In STEGR, the coincident gauge plays a special role by setting the affine connection $\Gamma^\alpha{}_{\mu\nu}$ to zero through a specific coordinate choice. However, this gauge choice explicitly breaks manifest diffeomorphism invariance because it selects a privileged coordinate system. This symmetry breaking is superficial, as it does not signal a fundamental loss of symmetry in the theory. In fact, the coincident gauge affects only a total derivative boundary term $B$ in the action, which does not influence the field equations or the dynamical evolution of the metric.

When a general coordinate transformation is applied, moving away from the coincident gauge, a non-trivial but purely inertial connection reappears as dictated by the transformation law (5.2.1). Despite this, the STEGR Lagrangian, constructed from the non-metricity scalar $\overset{\diamond}{Q}$, remains invariant under diffeomorphisms because it is built from covariant geometric objects. Therefore, while the coincident gauge hides diffeomorphism invariance at the level of coordinates, the full theory remains generally covariant, and the physical content is unaffected by the gauge choice.

It is important to remark that STEGR is not a gauge theory by definition. However, it can be formulated in a way that admits a gauge-like structure. This can be achieved by exploiting the freedom in the choice of spacetime isometries and by imposing the coincident gauge, which renders the affine connection a pure-gauge object. In this formulation, the connection carries no independent physical degrees of freedom and can be eliminated everywhere. As a consequence, at each spacetime point it is always possible to introduce a locally inertial frame in which the local dynamics reduces to that of Special Relativity. In this sense, the Strong Equivalence Principle is recovered, expressing the possibility of locally eliminating gravitational effects through an appropriate choice of reference frame.

Therefore, the coincident gauge embodies and preserves the Equivalence Principle of General Relativity.

Consequently, although it is not correct to regard STEGR as a gauge theory in the usual sense, it can be interpreted as admitting a gauge-theoretic structure when the Strong Equivalence Principle itself is viewed as a gauge symmetry, made manifest.

Moreover, we can formulate the STG gravitational coupling prescription, which can be schematized as follows

| STG | | GR |
|---|---|---|
| $\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$ | $\longleftrightarrow$ | $\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$ |
| $\partial_\mu \longrightarrow \overset{\diamond}{\nabla}_\mu$ | | |

## 5.3 STEGR Field Equations

STEGR embodies Palatini's idea, where metric $g_{\mu\nu}$ and affine connection $\Gamma^\rho{}_{\mu\nu}$ are two separated dynamical structures.

### 5.3.1 Variation with respect to the metric

The STEGR field equations are obtained by varying the STEGR action with respect to the metric tensor, namely,

$$\delta S_{STEGR} = \int d^4x \left[ \frac{c^4}{16\pi G} \delta(\sqrt{-g}\overset{\diamond}{Q}) + \delta(\sqrt{-g}\mathcal{L}_m) \right] = 0, \tag{5.3.1}$$

hence

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\overset{\diamond}{Q} + \sqrt{-g}\frac{\delta\overset{\diamond}{Q}}{\delta g^{\mu\nu}} = -\frac{16\pi G}{c^4}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}. \tag{5.3.2}$$

Firstly, we will focus on the $\delta\overset{\diamond}{Q}$ term. Let us write explicitly this term using eq.(5.3.13):

$$\delta\overset{\diamond}{Q} = -\frac{1}{4}\delta(\overset{\diamond}{Q}_\mu\overset{\diamond}{Q}^\mu) + \frac{1}{2}\delta(\overset{\diamond}{Q}_\mu\overset{\diamond}{\tilde{Q}}{}^\mu) - \frac{1}{4}\delta(\overset{\diamond}{Q}_{\mu\nu\sigma}\overset{\diamond}{Q}{}^{\mu\nu\sigma}) - \frac{1}{2}\delta(\overset{\diamond}{Q}_{\mu\nu\sigma}\overset{\diamond}{Q}{}^{\nu\mu\sigma}), \tag{5.3.3}$$

thus, we need to compute the variation of these four terms. To do so, we firstly calculate the variation of the non-metricity tensor, which will be helpful later.

$$\begin{aligned} \delta\overset{\diamond}{Q}_{\lambda\mu\nu} = \delta(\overset{\diamond}{\nabla}_\lambda g_{\mu\nu}) &= \overset{\circ}{\nabla}_\lambda \delta g_{\mu\nu} - \overset{\diamond}{L}{}^\alpha{}_{\lambda\mu}\delta g_{\alpha\nu} - \overset{\diamond}{L}{}^\alpha{}_{\lambda\nu}\delta g_{\mu\alpha} \\ &= \overset{\circ}{\nabla}_\lambda \delta g_{\mu\nu} - 2\overset{\diamond}{L}{}^\alpha{}_{\lambda(\mu}\delta g_{\alpha)\nu}, \end{aligned} \tag{5.3.4}$$

where start by assuming the gauge coincidence to simplify the calculations and the fact that the STG covariant derivative $\overset{\diamond}{\nabla}$ (5.2.10) could be written as the LC covariant derivative $\overset{\circ}{\nabla}$

plus the disformation correction.
Let us start with a useful variation:

$$\begin{aligned}\delta\overset{\diamond}{Q}_\lambda = \delta(g^{\mu\nu}\overset{\diamond}{Q}_{\lambda\mu\nu}) &= \overset{\diamond}{Q}_{\lambda\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta\overset{\diamond}{Q}_{\lambda\mu\nu} \\ &= \overset{\diamond}{Q}_{\lambda\mu\nu}(-g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}) + g^{\mu\nu}(\overset{\circ}{\nabla}_\lambda\delta g_{\mu\nu} - 2\overset{\diamond}{L}^{\alpha}{}_{\lambda(\mu}\delta g_{\alpha)\nu}) \\ &= -\overset{\diamond}{Q}_\lambda{}^{\mu\nu}\delta g_{\mu\nu} + g^{\mu\nu}\overset{\circ}{\nabla}_\lambda\delta g_{\mu\nu} - 2\overset{\diamond}{L}^{\mu}{}_\lambda{}^\nu\delta g_{\mu\nu}, \end{aligned} \tag{5.3.5}$$

where we used the variation of the metric tensor $\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}$. The next useful term is

$$\delta\overset{\diamond}{\tilde{Q}} = \delta(g^{\lambda\mu}\overset{\diamond}{Q}_{\lambda\mu\nu}) = -\overset{\diamond}{Q}^{\lambda\mu}{}_\nu\delta g_{\lambda\mu} + g^{\lambda\mu}\overset{\circ}{\nabla}_\lambda\delta g_{\mu\nu} - \overset{\diamond}{L}^{\alpha\mu}{}_\mu\delta g_{\alpha\nu} - \overset{\diamond}{L}^{\alpha\mu}{}_\nu\delta g_{\alpha\mu}. \tag{5.3.6}$$

The first term of the variation (5.3.3) is

$$\begin{aligned}\delta(\overset{\diamond}{Q}_\mu\overset{\diamond}{Q}^\mu) = \delta(g^{\mu\alpha}\overset{\diamond}{Q}_\alpha\overset{\diamond}{Q}_\mu) &= \overset{\diamond}{Q}_\alpha\overset{\diamond}{Q}_\mu\delta g^{\mu\alpha} + 2\overset{\diamond}{Q}^\mu\delta\overset{\diamond}{Q}_\mu \\ &= -g^{\alpha\beta}g^{\rho\mu}\delta g_{\beta\rho}\overset{\diamond}{Q}_\alpha\overset{\diamond}{Q}_\mu + 2\overset{\diamond}{Q}^\mu\delta\overset{\diamond}{Q}_\mu \\ &= -\overset{\diamond}{Q}^\mu\overset{\diamond}{Q}^\nu\delta g_{\mu\nu} + 2\overset{\diamond}{Q}^\mu\delta\overset{\diamond}{Q}_\mu \qquad \text{(inserting eq.(5.3.5))} \\ &= -\overset{\diamond}{Q}^\mu\overset{\diamond}{Q}^\nu\delta g_{\mu\nu} + 2\overset{\diamond}{Q}^\lambda(-\overset{\diamond}{Q}_\lambda{}^{\mu\nu}\delta g_{\mu\nu} + g^{\mu\nu}\overset{\circ}{\nabla}_\lambda\delta g_{\mu\nu} - 2\overset{\diamond}{L}^\mu{}_\lambda{}^\nu\delta g_{\mu\nu}) \\ &= -\overset{\diamond}{Q}^\mu\overset{\diamond}{Q}^\nu\delta g_{\mu\nu} - 2\overset{\diamond}{Q}^\lambda\overset{\diamond}{Q}_\lambda{}^{\mu\nu}\delta g_{\mu\nu} + 2\overset{\diamond}{Q}^\lambda g^{\mu\nu}\overset{\circ}{\nabla}_\lambda\delta g_{\mu\nu} - 4\overset{\diamond}{Q}^\lambda\overset{\diamond}{L}^\mu{}_\lambda{}^\nu\delta g_{\mu\nu} \\ &= 2\overset{\circ}{\nabla}_\lambda(\overset{\diamond}{Q}^\lambda g^{\mu\nu}\delta g_{\mu\nu}) - \left[2\overset{\diamond}{Q}^\lambda\overset{\diamond}{Q}_\lambda{}^{\mu\nu} + 2\overset{\circ}{\nabla}_\lambda(\overset{\diamond}{Q}^\lambda g^{\mu\nu}) + 4\overset{\diamond}{Q}^\lambda\overset{\diamond}{L}^\mu{}_\lambda{}^\nu + \overset{\diamond}{Q}^\mu\overset{\diamond}{Q}^\nu\right]\delta g_{\mu\nu}, \end{aligned} \tag{5.3.7}$$

where we used the Leibniz rule in the last line.
The second term of the variation is

$$\begin{aligned}\delta(\overset{\diamond}{Q}_\mu\overset{\diamond}{\tilde{Q}}^\mu) &= \overset{\diamond}{Q}^\mu\delta\overset{\diamond}{\tilde{Q}}_\mu + \overset{\diamond}{\tilde{Q}}^\mu\delta\overset{\diamond}{Q}_\mu - \overset{\diamond}{Q}^\mu\overset{\diamond}{\tilde{Q}}^\nu\delta g_{\mu\nu} \\ &= \overset{\circ}{\nabla}_\lambda(\overset{\diamond}{Q}^\nu g^{\lambda\mu}\delta g_{\mu\nu}) + \overset{\circ}{\nabla}_\lambda(\overset{\diamond}{\tilde{Q}}^\lambda g^{\mu\nu}\delta g_{\mu\nu}) + \left[\overset{\diamond}{Q}^\lambda\overset{\diamond}{Q}_\lambda{}^{\mu\nu} + \overset{\circ}{\nabla}_\lambda(\overset{\diamond}{Q}^\nu g^{\lambda\mu}) + \overset{\diamond}{Q}^\nu g^{\lambda\gamma}\overset{\diamond}{L}^\mu{}_{\lambda\gamma}\right. \\ &\qquad \left. + \overset{\diamond}{Q}^\gamma g^{\lambda\mu}\overset{\diamond}{L}^\nu{}_{\lambda\gamma} + \overset{\diamond}{\tilde{Q}}^\lambda\overset{\diamond}{Q}_\lambda{}^{\mu\nu} + \overset{\circ}{\nabla}_\lambda(\overset{\diamond}{\tilde{Q}}^\lambda g^{\mu\nu} + 2\overset{\diamond}{\tilde{Q}}^\lambda g^{\mu\gamma}\overset{\diamond}{L}^\nu{}_{\lambda\gamma} + \overset{\diamond}{Q}^\mu\overset{\diamond}{\tilde{Q}}^\nu\right]\delta g_{\mu\nu}. \end{aligned} \tag{5.3.8}$$

The third term is

$$\begin{aligned}
\delta(\overset{\diamond}{Q}_{\mu\nu\sigma}\overset{\diamond}{Q}^{\mu\nu\sigma}) &= \delta(g^{\mu\alpha}g^{\nu\beta}g^{\sigma\gamma}\overset{\diamond}{Q}_{\mu\nu\sigma}\overset{\diamond}{Q}_{\alpha\beta\gamma}) \\
&= 2\overset{\diamond}{Q}^{\mu\nu\sigma}\delta\overset{\diamond}{Q}_{\mu\nu\sigma} - \overset{\diamond}{Q}^{\lambda\beta\gamma}\overset{\diamond}{Q}^{\rho}{}_{\beta\gamma}\delta g_{\lambda\rho} - \overset{\diamond}{Q}^{\mu\rho\sigma}\overset{\diamond}{Q}_{\mu}{}^{\lambda}{}_{\sigma}\delta g_{\rho\lambda} - \overset{\diamond}{Q}^{\alpha\beta\rho}\overset{\diamond}{Q}_{\alpha\beta}{}^{\lambda}\delta g_{\lambda\rho} \\
&= 2\overset{\diamond}{Q}^{\mu\nu\sigma}\big[\overset{\circ}{\nabla}_{\mu}\delta g_{\nu\sigma} - 2\overset{\diamond}{L}^{\alpha}{}_{\mu(\nu}\delta g_{\alpha)\sigma}\big] - \overset{\diamond}{Q}^{\lambda\beta\gamma}\overset{\diamond}{Q}^{\rho}{}_{\beta\gamma}\delta g_{\lambda\rho} - 2\overset{\diamond}{Q}^{\alpha\beta\rho}\overset{\diamond}{Q}_{\alpha\beta}{}^{\lambda}\delta g_{\lambda\rho} \\
&= 2\overset{\circ}{\nabla}_{\mu}(\overset{\diamond}{Q}^{\mu\nu\sigma}\delta g_{\nu\sigma}) - \big[2\overset{\circ}{\nabla}_{\mu}\overset{\diamond}{Q}^{\mu\nu\sigma} + 4\overset{\diamond}{Q}^{\mu\alpha\sigma}\overset{\diamond}{L}^{\nu}{}_{\mu\alpha} + \overset{\diamond}{Q}^{\nu\beta\gamma}\overset{\diamond}{Q}^{\sigma}{}_{\beta\gamma} + 2\overset{\diamond}{Q}^{\alpha\beta\sigma}\overset{\diamond}{Q}_{\alpha\beta}{}^{\nu}\big]\delta g_{\nu\sigma}.
\end{aligned} \tag{5.3.9}$$

Finally, the last term of the variation is

$$\begin{aligned}
\delta(\overset{\diamond}{Q}_{\mu\nu\sigma}\overset{\diamond}{Q}^{\nu\mu\sigma}) &= \delta(g^{\alpha\nu}g^{\beta\mu}g^{\gamma\sigma}\overset{\diamond}{Q}_{\mu\nu\sigma}\overset{\diamond}{Q}_{\alpha\beta\gamma}) \\
&= 2\overset{\diamond}{Q}^{\nu\mu\sigma}\delta\overset{\diamond}{Q}_{\mu\nu\sigma} - \big[\overset{\diamond}{Q}^{\beta\rho\gamma}\overset{\diamond}{Q}^{\lambda}{}_{\beta\gamma} + \overset{\diamond}{Q}^{\rho\alpha\gamma}\overset{\diamond}{Q}_{\alpha}{}^{\lambda}{}_{\gamma} + \overset{\diamond}{Q}^{\beta\alpha\rho}\overset{\diamond}{Q}_{\alpha\beta}{}^{\lambda}\big]\delta g_{\lambda\rho} \\
&= 2\overset{\circ}{\nabla}_{\mu}(\overset{\diamond}{Q}^{\nu\mu\sigma}\delta g_{\nu\sigma}) - \big[2\overset{\circ}{\nabla}_{\mu}\overset{\diamond}{Q}^{\lambda\mu\rho} + \overset{\diamond}{Q}^{\beta\rho\gamma}\overset{\diamond}{Q}^{\lambda}{}_{\beta\gamma} + \overset{\diamond}{Q}^{\rho\alpha\gamma}\overset{\diamond}{Q}_{\alpha}{}^{\lambda}{}_{\gamma} + \overset{\diamond}{Q}^{\beta\alpha\rho}\overset{\diamond}{Q}_{\alpha\beta}{}^{\lambda}\big]\delta g_{\lambda\rho}.
\end{aligned} \tag{5.3.10}$$

So far, we have computed all the terms of $\sqrt{-g}\delta\overset{\diamond}{Q}$ of (5.3.2), and we can summarize them in the following way:

$$\sqrt{-g}\delta\overset{\diamond}{Q} = \sqrt{-g}\Big[-\frac{1}{4}(5.3.7) + \frac{1}{2}(5.3.8) - \frac{1}{4}(5.3.9) - \frac{1}{2}(5.3.10)\Big]. \tag{5.3.11}$$

Instead, the other term of the left hand side of (5.3.2) is

$$\delta\sqrt{-g}\overset{\diamond}{Q} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\overset{\diamond}{Q}. \tag{5.3.12}$$

Using eqs.(5.3.11) and (5.3.12) we can define the quantity

$$\frac{1}{\sqrt{-g}}\overset{\diamond}{q}_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\overset{\diamond}{Q})}{\partial g^{\mu\nu}} + \frac{1}{2}g_{\mu\nu}\overset{\diamond}{Q}. \tag{5.3.13}$$

In order to rewrite the field equations in a more meaningful way, we introduce the *non-metricity conjugate* $\overset{\diamond}{P}{}^{\alpha}{}_{\mu\nu}$ [17] as

$$\begin{aligned}\overset{\diamond}{P}{}^{\alpha}{}_{\mu\nu} &= \frac{1}{2\sqrt{-g}}\frac{\partial(\sqrt{-g}\overset{\diamond}{Q})}{\partial \overset{\diamond}{Q}{}_{\alpha}{}^{\mu\nu}} \\ &= \frac{1}{4}\overset{\diamond}{Q}{}^{\alpha}{}_{\mu\nu} - \frac{1}{4}\overset{\diamond}{Q}{}_{(\mu}{}^{\alpha}{}_{\nu)} - \frac{1}{4}g_{\mu\nu}\overset{\diamond}{Q}{}^{\alpha\beta}{}_{\beta} + \frac{1}{4}\left(\overset{\diamond}{Q}{}_{\beta}{}^{\beta\alpha}g_{\mu\nu} + \frac{1}{2}\delta^{\alpha}{}_{(\mu}\overset{\diamond}{Q}{}_{\nu)}{}^{\beta}{}_{\beta}\right),\end{aligned} \tag{5.3.14}$$

such that we the non-metricity scalar can be written as $\overset{\diamond}{Q} = \overset{\diamond}{Q}{}_{\alpha\mu\nu}\overset{\diamond}{P}{}^{\alpha\mu\nu}$.
In this way, the STEGR field equations (5.3.2)

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\overset{\diamond}{Q} + \sqrt{-g}\frac{\delta\overset{\diamond}{Q}}{\delta g^{\mu\nu}} = -\frac{16\pi G}{c^4}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$$

can be finally rewritten as

$$\boxed{\overset{\diamond}{G}{}_{\mu\nu} = \frac{2}{\sqrt{-g}}\overset{\circ}{\nabla}{}_{\alpha}\left(\sqrt{-g}\overset{\diamond}{P}{}^{\alpha}{}_{\mu\nu}\right) - \frac{1}{\sqrt{-g}}\overset{\diamond}{q}{}_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\overset{\diamond}{Q} = \frac{8\pi G}{c^4}\mathfrak{T}_{\mu\nu}}, \tag{5.3.15}$$

where $\overset{\diamond}{G}{}_{\mu\nu}$ is the STEGR Einstein tensor and

$$\mathfrak{T}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \tag{5.3.16}$$

is the usual energy-momentum tensor for matter.

### 5.3.2 Equivalence with GR field equations

In STEGR we can rewrite the second Bianchi identity (4.3.54) as

$$\partial_{\lambda}R^{\alpha}{}_{\beta\mu\nu} + \partial_{\nu}R^{\alpha}{}_{\beta\lambda\mu} + \partial_{\mu}R^{\alpha}{}_{\beta\nu\lambda} = 0, \tag{5.3.17}$$

where we already made use of the coincident gauge and the curvature tensor is

$$R^{\alpha}{}_{\beta\mu\nu} = \overset{\circ}{R}{}^{\alpha}{}_{\beta\mu\nu} + \overset{\diamond}{\mathcal{L}}{}^{\alpha}{}_{\beta\mu\nu} = 0 \tag{5.3.18}$$

with

$$\overset{\diamond}{\mathcal{L}}^{\alpha}{}_{\beta\mu\nu} = \overset{\circ}{\nabla}_{\mu}\overset{\diamond}{L}^{\alpha}{}_{\beta\nu} - \overset{\circ}{\nabla}_{\nu}\overset{\diamond}{L}^{\alpha}{}_{\beta\mu} + \overset{\diamond}{L}^{\alpha}{}_{\sigma\mu}\overset{\diamond}{L}^{\sigma}{}_{\beta\nu} - \overset{\diamond}{L}^{\alpha}{}_{\sigma\nu}\overset{\diamond}{L}^{\sigma}{}_{\beta\mu} \tag{5.3.19}$$

such that $\overset{\diamond}{\mathcal{L}}^{\alpha}{}_{\beta\mu\nu} = -\overset{\diamond}{\mathcal{L}}^{\beta}{}_{\alpha\mu\nu}$ and $\overset{\diamond}{\mathcal{L}}^{\alpha}{}_{\beta\mu\nu} = -\overset{\diamond}{\mathcal{L}}^{\alpha}{}_{\beta\nu\mu}$. It follows that eq.(5.3.17) becomes

$$\partial_{\lambda}\overset{\circ}{R}^{\lambda}{}_{\beta\mu\nu} + \partial_{\mu}\overset{\circ}{R}^{\lambda}{}_{\beta\nu\lambda} + \partial_{\nu}\overset{\circ}{R}^{\lambda}{}_{\beta\lambda\mu} + \partial_{\lambda}\overset{\diamond}{\mathcal{L}}^{\lambda}{}_{\beta\mu\nu} + \partial_{\mu}\overset{\diamond}{\mathcal{L}}^{\lambda}{}_{\beta\nu\lambda} + \partial_{\nu}\overset{\diamond}{\mathcal{L}}^{\lambda}{}_{\beta\lambda\mu} = 0. \tag{5.3.20}$$

Following the same procedure as in Sec.(4.3.4.1), we obtain

$$\boxed{\overset{\circ}{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\overset{\circ}{R} = -\overset{\diamond}{\mathcal{L}}_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\overset{\diamond}{\mathcal{L}}}, \tag{5.3.21}$$

where $\overset{\diamond}{\mathcal{L}}_{\mu\nu} = \overset{\diamond}{\mathcal{L}}^{\alpha}{}_{\mu\alpha\nu}$ and $\overset{\diamond}{\mathcal{L}} = \overset{\diamond}{\mathcal{L}}^{\mu}{}_{\mu}$, showing the equivalence with the GR field equations. Furthermore, we have

$$\boxed{\overset{\diamond}{\mathcal{L}}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\overset{\diamond}{\mathcal{L}} = 0}, \tag{5.3.22}$$

which are the STEGR field equations.
To retrieve a full equivalence, we must prove that the STEGR equations (5.3.22) are equal to those found previously with the minimization of the action, i.e. (5.3.15). Hence,

$$\begin{gathered}\overset{\diamond}{G}_{\mu\nu} = \frac{2}{\sqrt{-g}}\overset{\circ}{\nabla}_{\alpha}\big(\sqrt{-g}\overset{\diamond}{P}^{\alpha}{}_{\mu\nu}\big) - \frac{1}{\sqrt{-g}}\overset{\diamond}{q}_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\overset{\diamond}{Q} = 0\\ \Updownarrow \\ \overset{\diamond}{\mathcal{L}}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\overset{\diamond}{\mathcal{L}} = 0\end{gathered}$$

We will start from the first one. Firstly, recall the following identity

$$\frac{\partial_{\alpha}\sqrt{-g}}{\sqrt{-g}} = \overset{\circ}{\Gamma}^{\sigma}{}_{\alpha\sigma} = -\overset{\diamond}{L}^{\sigma}{}_{\alpha\sigma} = \frac{1}{2}\overset{\diamond}{Q}_{\alpha}. \tag{5.3.23}$$

Inserting eq.(5.3.13) and (5.3.14) in (5.3.15), we obtain

$$
\begin{aligned}
&\overbrace{\frac{2}{\sqrt{-g}}\big(\partial_\alpha\sqrt{-g}\big)}^{\overset{\diamond}{Q}_\alpha} P^\alpha{}_{\mu\nu} + \frac{2}{\sqrt{-g}}\sqrt{-g}\partial_\alpha P^\alpha{}_{\mu\nu} - \frac{1}{\sqrt{-g}}\overset{\diamond}{q}_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\overset{\diamond}{Q} = 0 \\
&\frac{1}{2}\overset{\diamond}{L}^\alpha{}_{\mu\nu}\overset{\diamond}{Q}_\alpha - \frac{1}{4}g_{\mu\nu}\overset{\diamond}{Q}_\alpha\big(\overset{\diamond}{Q}^\alpha - \overset{\diamond}{\tilde{Q}}^\alpha\big) + \cancel{\frac{1}{4}\overset{\diamond}{Q}_\mu\overset{\diamond}{Q}_\nu} + 2\partial_\alpha P^\alpha{}_{\mu\nu} - \frac{1}{4}\big(\underline{2\overset{\diamond}{Q}_{\alpha\beta\mu}\overset{\diamond}{Q}^{\alpha\beta}{}_\nu} - \overset{\diamond}{Q}_{\mu\alpha\beta}\overset{\diamond}{Q}_\nu{}^{\alpha\beta}\big) \\
&\qquad + \frac{1}{4}\big(\underset{\sim\sim\sim}{2\overset{\diamond}{Q}_\alpha\overset{\diamond}{Q}^\alpha{}_{\mu\nu}} - \cancel{\overset{\diamond}{Q}_\mu\overset{\diamond}{Q}_\nu}\big) + \frac{1}{2}\big(\underline{\overset{\diamond}{Q}_{\alpha\beta\mu}\overset{\diamond}{Q}^{\beta\alpha}{}_\nu} - \underset{\sim\sim\sim}{\overset{\diamond}{\tilde{Q}}_\alpha\overset{\diamond}{Q}^\alpha{}_{\mu\nu}}\big) + \frac{1}{2}g_{\mu\nu}\overset{\diamond}{Q} = 0 \\
&2\partial_\alpha P^\alpha{}_{\mu\nu} + \frac{1}{2}\overset{\diamond}{Q}_{\alpha\mu\nu}\big(\overset{\diamond}{Q}^\alpha - \overset{\diamond}{\tilde{Q}}^\alpha\big) + \frac{1}{2}\overset{\diamond}{Q}_{\alpha\beta\mu}\big(\overset{\diamond}{Q}^{\beta\alpha}{}_\nu - \overset{\diamond}{Q}^{\alpha\beta}{}_\nu\big) - \frac{1}{2}g_{\mu\nu}\overset{\circ}{\nabla}_\alpha\big(\overset{\diamond}{Q}^\alpha - \overset{\diamond}{\tilde{Q}}^\alpha\big) \\
&\qquad + \frac{1}{2}g_{\mu\nu}\partial_\alpha\big(\overset{\diamond}{Q}^\alpha - \overset{\diamond}{\tilde{Q}}^\alpha\big) + \frac{1}{4}\overset{\diamond}{Q}_{\mu\alpha\beta}\overset{\diamond}{Q}_\nu{}^{\alpha\beta} + \frac{1}{2}\overset{\diamond}{L}^\alpha_{\mu\nu}\overset{\diamond}{Q}_\alpha + \frac{1}{2}g_{\mu\nu}\overset{\diamond}{Q} = 0.
\end{aligned}
\tag{5.3.24}
$$

In the last step we have rewritten the $-\frac{1}{4}g_{\mu\nu}\overset{\diamond}{Q}_\alpha\big(\overset{\diamond}{Q}^\alpha\overset{\diamond}{\tilde{Q}}^\alpha\big)$ term using

$$
\partial_\alpha\overset{\diamond}{Q}^\alpha = \overset{\circ}{\nabla}_\alpha\overset{\diamond}{Q}^\alpha - \frac{1}{2}\overset{\diamond}{Q}_\alpha\overset{\diamond}{Q}^\alpha \implies \overset{\diamond}{Q}_\alpha\big(\overset{\diamond}{Q}^\alpha - \overset{\diamond}{\tilde{Q}}^\alpha\big) = 2\partial_\alpha\big(\overset{\diamond}{Q}^\alpha - \overset{\diamond}{\tilde{Q}}^\alpha\big) - 2\nabla_\alpha\big(\overset{\diamond}{Q}^\alpha - \overset{\diamond}{\tilde{Q}}^\alpha\big). \tag{5.3.25}
$$

Now, we will retrieve the same expression as (5.3.24) starting from the second form of the STEGR field equations, i.e. eq.(5.3.22).
We start by writing explicitly $\overset{\diamond}{\mathcal{L}}_{\mu\nu}$ and $\overset{\diamond}{\mathcal{L}}$. Using eqs.(5.1.5), (3.1.9) and (3.1.11), i.e.

$$
\begin{aligned}
\overset{\diamond}{L}^\alpha{}_{\mu\alpha} &= -\frac{1}{2}\overset{\diamond}{Q}_\mu, \\
\overset{\diamond}{L}^\alpha{}_{\mu\nu} &= \frac{1}{2}\big(\overset{\diamond}{Q}^\alpha{}_{\mu\nu} - \overset{\diamond}{Q}_\mu{}^\alpha{}_\nu - \overset{\diamond}{Q}_\nu{}^\alpha{}_\mu,\big), \\
\overset{\diamond}{Q}_{\alpha\mu\nu} &= \overset{\diamond}{Q}_{\alpha\nu\mu},
\end{aligned}
$$

and contracting $\alpha$ and $\mu$ in (5.3.19), it follows that

$$
\begin{aligned}
\overset{\diamond}{\mathcal{L}}_{\mu\nu} &= \overset{\circ}{\nabla}_\alpha\overset{\diamond}{L}^\alpha{}_{\mu\nu} - \overset{\diamond}{\nabla}_\nu\overset{\diamond}{L}^\alpha{}_{\mu\alpha} + \overset{\diamond}{L}^\sigma{}_{\mu\nu}\overset{\diamond}{L}^\alpha{}_{\sigma\alpha} - \overset{\diamond}{L}^\sigma{}_{\mu\alpha}\overset{\diamond}{L}^\alpha{}_{\sigma\nu} \\
&= \overset{\circ}{\nabla}_\alpha\overset{\diamond}{L}^\alpha{}_{\mu\nu} + \frac{1}{2}\overset{\circ}{\nabla}_\nu\overset{\diamond}{Q}_\mu - \frac{1}{2}\overset{\diamond}{Q}_\alpha\overset{\diamond}{L}^\alpha{}_{\mu\nu} - \frac{1}{4}\big[\overset{\diamond}{Q}_\mu{}^\sigma{}_\alpha\overset{\diamond}{Q}_\nu{}^\alpha{}_\sigma + 2\overset{\diamond}{Q}^\alpha{}_{\sigma\nu}\big(\overset{\diamond}{Q}^\sigma{}_{\alpha\mu} - \overset{\diamond}{Q}_\alpha{}^\sigma{}_\mu\big)\big].
\end{aligned}
\tag{5.3.26}
$$

Hence,

$$
\begin{aligned}
\overset{\diamond}{\mathcal{L}} &= g^{\mu\nu}\overset{\diamond}{\mathcal{L}}_{\mu\nu} \\
&= \mathring{\nabla}_\alpha \overset{\diamond}{L}^{\alpha\nu}{}_{\nu} + \frac{1}{2}\mathring{\nabla}_\nu \overset{\diamond}{Q}^{\nu} - \frac{1}{2}\overset{\diamond}{Q}_\alpha \overset{\diamond}{L}^{\alpha\nu}{}_{\nu} - \frac{1}{4}\big[\overset{\diamond}{Q}^{\nu\sigma}{}_{\alpha}\overset{\diamond}{Q}_{\nu}{}^{\alpha}{}_{\sigma} + 2\overset{\diamond}{Q}^{\alpha}{}_{\sigma\nu}\big(\overset{\diamond}{Q}^{\sigma}{}_{\alpha}{}^{\nu} - \overset{\diamond}{Q}_{\alpha}{}^{\sigma\nu}\big)\big] \\
&= \mathring{\nabla}_\alpha\big(\frac{1}{2}\overset{\diamond}{Q}^{\alpha} - \overset{\diamond}{\tilde{Q}}^{\alpha}\big) + \frac{1}{2}\nabla_\alpha \overset{\diamond}{Q}^{\alpha} - \frac{1}{2}\overset{\diamond}{Q}_\alpha\big(\frac{1}{2}\overset{\diamond}{Q}^{\alpha} - \overset{\diamond}{\tilde{Q}}^{\alpha}\big) - \frac{1}{4}\overset{\diamond}{Q}^{\nu\sigma}{}_{\alpha}\overset{\diamond}{Q}_{\nu}{}^{\alpha}{}_{\sigma} - \frac{1}{2}\overset{\diamond}{Q}^{\alpha}{}_{\sigma\nu}\overset{\diamond}{Q}^{\sigma}{}_{\alpha}{}^{\nu} + \frac{1}{2}\overset{\diamond}{Q}^{\alpha}{}_{\sigma\nu}\overset{\diamond}{Q}_{\alpha}{}^{\sigma\nu} \\
&= \mathring{\nabla}_\alpha\big(\overset{\diamond}{Q}^{\alpha} - \overset{\diamond}{\tilde{Q}}^{\alpha}\big) - \overset{\diamond}{Q},
\end{aligned}
\tag{5.3.27}
$$

where we used the definition of non-metricity scalar (5.3.13).
Gathering together eq.(5.3.26) and (5.3.27) we can rewrite the STEGR field equations (5.3.22) as

$$
\begin{aligned}
\mathring{\nabla}_\alpha \overset{\diamond}{L}^{\alpha}{}_{\mu\nu} + \frac{1}{2}\mathring{\nabla}_\nu \overset{\diamond}{Q}_{\mu} - \frac{1}{2}\overset{\diamond}{Q}_\alpha \overset{\diamond}{L}^{\alpha}{}_{\mu\nu} - \frac{1}{4}\big[\overset{\diamond}{Q}_{\mu}{}^{\sigma}{}_{\alpha}\overset{\diamond}{Q}_{\nu}{}^{\alpha}{}_{\sigma} + 2\overset{\diamond}{Q}^{\alpha}{}_{\sigma\nu}\big(\overset{\diamond}{Q}^{\sigma}{}_{\alpha\mu} - \overset{\diamond}{Q}_{\alpha}{}^{\sigma}{}_{\mu}\big)\big] & \\
- \frac{1}{2}g_{\mu\nu}\big[\mathring{\nabla}_\alpha\big(\overset{\diamond}{Q}^{\alpha} - \overset{\diamond}{\tilde{Q}}^{\alpha}\big) - \overset{\diamond}{Q}\big] &= 0
\end{aligned}
\tag{5.3.28}
$$

Using now

$$
\overset{\diamond}{L}^{\alpha}{}_{\mu\nu} = 2\overset{\diamond}{P}^{\alpha}{}_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\big(\overset{\diamond}{Q}^{\alpha} - \overset{\diamond}{\tilde{Q}}^{\alpha}\big) - \frac{1}{4}\big(\delta^{\alpha}{}_{\mu}\overset{\diamond}{Q}_{\nu} + \delta^{\alpha}{}_{\nu}\overset{\diamond}{Q}_{\mu}\big),
\tag{5.3.29}
$$

we obtain

$$
\begin{aligned}
&2\partial_\alpha \overset{\diamond}{P}^{\alpha}{}_{\mu\nu} + \frac{1}{2}\overset{\diamond}{Q}_{\alpha\mu\nu}\big(\overset{\diamond}{Q}^{\alpha} - \overset{\diamond}{\tilde{Q}}^{\alpha}\big) + \frac{1}{2}g_{\mu\nu}\partial_\alpha\big(\overset{\diamond}{Q}^{\alpha} - \overset{\diamond}{\tilde{Q}}^{\alpha}\big) + \frac{1}{2}\overset{\diamond}{L}^{\sigma}{}_{\mu\nu}\overset{\diamond}{Q}_{\sigma} \\
&\quad + \frac{1}{4}\overset{\diamond}{Q}_{\mu}{}^{\sigma}{}_{\alpha}\overset{\diamond}{Q}_{\nu}{}^{\alpha}{}_{\sigma} + \frac{1}{2}\overset{\diamond}{Q}^{\alpha}{}_{\sigma\nu}\big(\overset{\diamond}{Q}^{\sigma}{}_{\alpha\mu} - \overset{\diamond}{Q}_{\alpha}{}^{\sigma}{}_{\mu}\big) - \frac{1}{2}g_{\mu\nu}\mathring{\nabla}_\alpha\big(\overset{\diamond}{Q}^{\alpha} - \overset{\diamond}{\tilde{Q}}^{\alpha}\big) + \frac{1}{2}g_{\mu\nu}\overset{\diamond}{Q} = 0
\end{aligned}
\tag{5.3.30}
$$

### 5.3.3 Variation with respect to the connection

We will now variate the STEGR action with respect to the connection. Hence, we start form eq.(5.3.1), and we obtain

$$
\frac{\delta\sqrt{-g}}{\delta\overset{\diamond}{\Gamma}^{\alpha}{}_{\mu\nu}}\overset{\diamond}{Q} + \sqrt{-g}\frac{\delta\overset{\diamond}{Q}}{\delta\overset{\diamond}{\Gamma}^{\alpha}{}_{\mu\nu}} = -\frac{16\pi G}{c^4}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\overset{\diamond}{\Gamma}^{\alpha}{}_{\mu\nu}}.
\tag{5.3.31}
$$

Since we are assuming the Palatini approach, the metric does not depend on the connection, thus the first term vanishes and it follows that

$$\frac{\delta \overset{\diamond}{Q}}{\delta \overset{\diamond}{\Gamma}{}^{\alpha}{}_{\mu\nu}} = -\frac{8\pi G}{c^4} H_{\alpha}{}^{\mu\nu}, \tag{5.3.32}$$

where

$$H_{\alpha}{}^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta \overset{\diamond}{\Gamma}{}^{\alpha}{}_{\mu\nu}} \tag{5.3.33}$$

is the STEGR *hypermomentum*. Recall that

$$\overset{\diamond}{Q}_{\lambda\mu\nu} = \overset{\diamond}{\nabla}_{\lambda} g_{\mu\nu} = \partial_{\lambda} g_{\mu\nu} - \overset{\diamond}{\Gamma}{}^{\alpha}{}_{\lambda\mu} g_{\alpha\nu} - \overset{\diamond}{\Gamma}{}^{\alpha}{}_{\lambda\nu} g_{\mu\alpha}, \tag{5.3.34}$$

hence,

$$\delta \overset{\diamond}{Q}_{\lambda\mu\nu} = -\delta \overset{\diamond}{\Gamma}{}^{\alpha}{}_{\lambda\mu} g_{\alpha\nu} - \delta \overset{\diamond}{\Gamma}{}^{\alpha}{}_{\lambda\nu} g_{\mu\alpha}. \tag{5.3.35}$$

The variation of the scalar torsion is then

$$\delta \overset{\diamond}{Q} = \frac{\delta \overset{\diamond}{Q}}{\delta \overset{\diamond}{Q}_{\lambda\mu\nu}} \delta \overset{\diamond}{Q}_{\lambda\mu\nu} = \overset{\diamond}{P}{}^{\lambda\mu\nu} \delta Q_{\lambda\mu\nu} = \overset{\diamond}{P}{}^{\lambda\mu\nu} \big( -\delta \overset{\diamond}{\Gamma}{}^{\alpha}{}_{\lambda\mu} g_{\alpha\nu} - \delta \overset{\diamond}{\Gamma}{}^{\alpha}{}_{\lambda\nu} g_{\mu\alpha} \big) = -2 \overset{\diamond}{P}{}^{\lambda\mu\nu} \delta \overset{\diamond}{\Gamma}{}^{\alpha}{}_{\lambda\mu} g_{\alpha\nu}. \tag{5.3.36}$$

Assuming an arbitrary connection variation and that the STG connection does not couple with matter, i.e. $H_{\alpha}{}^{\mu\nu} = 0$, we can retrieve the *connection field equations* integrating by parts

$$\frac{\delta S_{STEGR}}{\delta \Gamma^{\alpha}{}_{\lambda\mu}} = -2 \int d^4x \sqrt{-g} \overset{\diamond}{P}{}^{\lambda\mu\nu} g_{\alpha\nu} = 0,$$

$$\implies \boxed{\overset{\circ}{\nabla}_{\mu} \overset{\circ}{\nabla}_{\nu} \big( \sqrt{-g} \overset{\diamond}{P}_{\alpha}{}^{\mu\nu} \big) = 0}. \tag{5.3.37}$$

Note that they are trivially satisfied by the coincident gauge, meaning that they do not impose new dynamical constraints.

In STEGR we have that the total DoFs are enconded in the metric tensor, having 10 components from which we have to subtract 8 diffeomorphisms as in GR, having therefore again 2 DoFs as in GR. Here we have that the 4 diffeomorphisms of coordinates become the gauge diffeomorphism symmetries.

# Chapter 6

# Trinity of Gravity: its extensions, state of the art and open issues

As we have seen so far, the three theories of the Trinity have shown to be dynamically equivalent at Lagrangian level up to a boundary term, while it is also possible to retrieve the same field equations starting from the Second Bianchi identity. In the following, we will talk about the equivalence of the Trinity at solution level.
Moving forward, we will discuss about the extendend version of the Trinity and whether is possible to guarantee their equivalence. In the end, we will present the open problems and conceptual tensions that concern the Trinity of Gravity.

## 6.1 Solutions in Trinity of Gravity

We now briefly discuss the equivalence of the GR, TEGR and STEGR at solution level. Previously we obtained the same field equations for the Trinity, hence we expect that the same solutions have to be retieved if we impose the same symmetries. One of the simplest solution is the *Schwartzschild spacetime* [12, 18], i.e.

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2,$$

which is written in spherical coordinates $\{t, r, \theta, \varphi\}$ on the equatorial plane $\theta = \frac{\pi}{2}$.
To do so, it is convenient to use the $3+1$ slitting formalism of the metric and tetrads in spatial and temporal part to simplify the calculations and numerical analysis [27].

In the TEGR scenario, a spherically symmetric solution can be recovered adopting the tetrad formalism and the Weitzenbock gauge (see [28, 29] for a detailed discussion). As it is shown in [17], choosing a diagonal tetrad allows to prove that $\mathring{G}_{\mu\nu} = \hat{G}_{\mu\nu}$.

A similar discussion is done in STEGR. Here the coincident gauge $\overset{\diamond}{\nabla}_\mu = \partial_\mu$ and $\overset{\circ}{\Gamma}{}^\mu_{\alpha\beta} = \overset{\diamond}{L}{}^\mu{}_{\alpha\beta}$ is adopted, which makes even simpler the proof of $\overset{\circ}{G}_{\mu\nu} = \overset{\diamond}{G}_{\mu\nu}$.

It is important to remark that, in both TEGR and STEGR, the usual picture of a black hole as an object so massive that it bends spacetime into an infinite geometric well is no longer appropriate, since these theories are formulated on flat manifolds. In TEGR, the gravitational interaction manifests itself through the torsion of the tetrad fields, which induces rotations of the observer frame. Therefore, a black hole is not interpreted as a geometric well, but rather as a region where torsion becomes extremely large, formally divergent. In STEGR, instead, gravity is encoded in the non-metricity, which produces expansions and contractions of the observer laboratory. Consequently, the geometric effect of a black hole is understood as an extreme variation of measured distances, rather than a curvature-induced well.

Moreover, it is also possible to observe the validity of the *Birkoff theorem* [30] also in TEGR and STEGR, which states that *any spherically symmetric solution of the GR field equations in vacuum has to be static and asymptotically flat. In addition, the Schwartzschild solution is the unique solution satisfying these hypoteses.*

## 6.2 Considerations on the Extended Geometric Trinity of Gravity

Even if we will not discuss this theme in detail, it is correct to make the reader aware of the extensions of the dynamically equivalent representations of gravity. The study of modified Gravity theories have been proposed and motivated by discoveries in high energy physics, cosmology and astrophysics (see [31] for further details and references).

A modified theory of Gravity is the extension of the Einstein's theory by correcting the Einstein-Hilbert action with higher order terms in the curvature invariant, for example as $\overset{\circ}{R}{}^2$, $\overset{\circ}{R}_{\mu\nu}\overset{\circ}{R}{}^{\mu\nu}$, $\overset{\circ}{R}_{\mu\nu\rho\sigma}\overset{\circ}{R}{}^{\mu\nu\rho\sigma}$, and in some cases also terms contaning minimally and non-minimally coupled scalar fields. Nonetheless, we will consider the class of $f(\overset{\circ}{R})$ theories of gravity, in which the Ricci scalar is substituted with a general smooth function of $\overset{\circ}{R}$, namely [21]

$$S_{f(\overset{\circ}{R})} = \int d^4x \left[\sqrt{-g} f(\overset{\circ}{R}) + 2\chi \mathcal{L}_m\right].$$

By means of the same procedure, we can also extend the TEGR and STEGR theories [16]. It is important to remark that in general the equivalence between the three theories is not valid at the extended level anymore. The main problem is due to the fact the the equivalence holds only for theories which are linear in the scalar invariant, while the extension can exhibit further (and different) degrees of freedom. However, the equivalence at the extended level can be restored making use of a correct boundary term identification (see [21] for details).

### 6.2.1 Extended TEGR

In the extended TEGR scenario, we start by substituting the torsion scalar with a smooth function $f(\hat{T}, \tilde{B})$, namely

$$S_{f(\hat{T},\tilde{B})} = \int d^4x \big[e f(\hat{T}, \tilde{B}) + 2\chi \mathcal{L}_m\big], \tag{6.2.1}$$

where $\tilde{B}$ is the boundary term (4.3.39). It is now possible to recover the $f(\mathring{R})$ gravity if we impose the condition (4.3.38), i.e. $\mathring{R} = -\hat{T} - \tilde{B}$, obtaining

$$f(\hat{T}, \tilde{B}) = f(-\hat{T} - \tilde{B}) = f(\mathring{R}).$$

Moreover, it is also possible to retrieve gravitational waves in $f(\hat{T}, \tilde{B})$. In the weak field gravity, they coincide with those of $f(\mathring{R})$ gravity and exhibit three polarizations: the two standard of general relativity, plus and cross, which are purely transverse with two-helicity, massless tensor polarization modes, and an additional massive scalar mode with zero-helicity [32].

Clearly, if we consider $f(\hat{T}, \tilde{B}) = f(\hat{T})$, and in particular $f(\hat{T}) = \hat{T}$, we recover GR and the equivalence between GR and TEGR is restored.

### 6.2.2 Extended STEGR

In the case of extended STEGR, we write the action as

$$S_{f(\overset{\diamond}{Q},B)} = \int d^4x \big[\sqrt{-g} f(\overset{\diamond}{Q}, B) + 2\chi \mathcal{L}_m\big],$$

where $B$ is the boundary term defined in eq.(5.1.6). As before, $f(\mathring{R})$ gravity is recovered by imposing the identity (5.1.8), i.e. $\mathring{R} = \overset{\diamond}{Q} - B$, which leads to

$$f(\overset{\diamond}{Q}, B) = f(\overset{\diamond}{Q} - B) = f(\mathring{R}).$$

In the special case $f(\overset{\diamond}{Q}, B) = f(\overset{\diamond}{Q})$, we will obtain second order field equations as in the $f(\hat{T})$ gravity.

### 6.2.3 Comments on Renormalization

Both standard TEGR and STEGR are non-renormalizable in the same way that General Relativity is, which means they cannot be treated as a complete quantum field theory using standard techniques.
Roughly speaking, the non-renormalizability of Gravity means that when trying to quantize it, the theory becomes unpredictable at high energies, as it requires an infinite number of counterterms to fix the divergences.

However, it has been shown that a possible attempt to renormalize Gravity could be done by using extended theories. In fact, in order to renormalize matter fields in curved space-time, higher-order curvature terms as $R^2, R^{\mu\nu}R_{\mu\nu}, R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ must be considered within the effective action [33]. Furthermore, gravity theories with quadratic curvature terms turn out to be renormalizable and asymptotically free in the metric framework [34, 35]. However, higher-derivative gravity theories generically contain a massive spin-2 ghost appearing as an additional pole in the propagator. The position of this ghost pole is gauge independent, but its presence leads to potential violations of unitarity, and its physical interpretation remains a subject of ongoing debate [36, 37, 38].

## 6.3 Motivations for studying Teleparallel Gravity

First of all, since we have constructed and discussed the Teleparallel Gravity, we have all the ingredients to fully comprehend why we are interested in studying this theory [16]. We could have presented this argument at the beginning of our journey, but doing so would have required anticipating results that are formally introduced later. In this section, we will restrict ourselves to TEGR.

## Gauge theory for Gravity

We have seen that TEGR is a theory equivalent to GR. It allowed us to treat the gravitational interaction from a different perspective: the concept of curvature in GR is replaced by torsion, while the geometry by the force. The first point which changes the game is the fact that TEGR is fomulated as a gauge theory for the spacetime translations. This description not only explains that the source of gravitation is the energy-momentum, which is the Noether current for those translations, but allows us to treat the gravitation by means of a gauge theory, as already happens with the other known interactions.

## Gravitational energy-momentum density

In all the field theories it possible to define a local energy-momentum density. This is a quantity described by a tensor, hence is frame-independent. It would then be natural to expect that also the gravitational field possesses such thing. Unfortunately, in GR there is no tensorial expression for the gravitational energy-momentum density. The reason of that lies on the impossibility to separate the gravitational and inertial effects, which are both contained in the GR spin connection. Hence, the gravitational energy-momentum density of GR has to include both a contribution from gravitation and inertials effect, and since the latter are non-tensorial, then the energy-momentum density will be a non-tensorial object.
This is not a malfunction of the theory, but rather a consequence of the Equivalence Principle. In fact, it is always possible to locally neglect gravitation effects choosing an appropriate reference system.

On the other side, in TEGR the inertial effects are described by a Lorentz connection, while the gravitation is represented by a translational gauge potential, which is the non-trivial part of the tetrad. Hence, this theory provides a separation between inertial effects and gravitational ones and, as a consequence, it is possible to write an expresson for the energy-momentum density. Obviously, in order to be a true tensor, it has to include only the gravitational term, which involves only a covariant conservation.
This discovery tells us that the difficulties encountered in GR to define an energy-momentum density, are strictly related to the GR framework and not to the nature of the gravitational interactoion.

## Spin-2 field–Gravity coupling

Fundamental spin-2 fields present inconsistencies when coupled to gravity [39, 40]. This is caused by the fact that the divergence identities satisfied by the field equations of a spin-2 field in a free theory (Minkowski spacetime) are no longer true when we introduce a coupling to gravity: the derivatives are transformed in covariant derivatives by the coupling prescription, which introduce terms proportional to the curvature due to their non-commutativity.

If we instead describe a spin-2 field with tetrads, we pass from a rank-2 symmetric tensor $\psi_{\mu\nu}$ to a 1-form $\psi^a{}_\mu$, which takes value in the translational Lie algebra, while its interpretation goes from a linear perturbation of the metric to one of the tetrads.
In absence of gravitation the Teleparallel approach to the spin-2 field correponds to the GR approach, but in the presence of gravitation it differs substantially. The reason is that the latin index of the translational-valued field $\psi^a{}_\mu$ is not an ordinary vector index, but a gauge index. As such, it is irrelevant for the gravitational coupling prescription because it labels an internal symmetry (translations) and does not couple to the gravitational connection. This way, we give rise to a spin-2 field theory, where the perturbation of the tetrad $\psi^a{}_\mu$ is a gauge field, analogous to the electromagnetic potential $A_\mu$.
This theory is shown to be gauge invariant and local Lorentz invariant, and preserves the duality symmetry of the free theory. Moreover, there are no constraints on the spacetime geometry thanks to the spin connection not having inertia. This leads to a tetrad-based gravitationally coupled spin-2 theory fully consistent.

## Gravitation and Quantum Field Theory

As we know, General Relativity and Quantum Mechanics are not consistent with each other. GR is based on the Equivalence Principle, whose strong version establishes the local equivalence between gravitation and inertia. This principle makes use of *ideal observers*, which are time-like curve, i.e. wordlines [41]. Such curve represents locally, in well-chosen coordinates, a point-like object in 3-space evolving in the timelike 4th direction.
Hence, along the trajectory of a point-like ideal observer, gravity can be neglected by choosing an appropriate reference system. Remember that, mathematically, the Levi-Civita connection can be nullified along that curve.
Quantum Mechanics has the Uncertainty Principle as foundational principle, and it is nonlocal: a test particle does not follow a given trajectory, but infinitely many with different probabilities. Thus, QM requires instead *real observers*, i.e. objects extended in space, which by definition intersect congruence of curves, described by the deviation geodesics equation and depend on the spacetime curvature. For this reason, it seems that the GR framework is incompatible with QM. Therefore, a quantum version of EP is impossible.

On the other hand, as we have already seen, the Teleparallel Gravity does not require the EP as foundational principle. As a consequence, if we consider TG then we can not consider the Equivalence Principle. Obviously, this step does not eliminate the inconsistencies between gravitation and QM, but surely it provides a better framework to deal with these problems.

Moreover, this framework appears to provide a promising route toward the quantization of Gravity. Since gravitational effects are encoded in a translational gauge potential that cannot be made to vanish locally, unlike the spin connection in GR, it constitutes a natural field variable to be quantized in approaches to Quantum Gravity.

## 6.4 Considerations and implications of the fundamental role of the Equivalence Principle

In the previous chapter, we explored the formulations of TEGR and STEGR and demonstrated their dynamical equivalence. This allowed us to conclude that GR, TEGR, and STEGR are equivalent at the Lagrangian level, in the field equations derived from the second Bianchi identity, and in their respective solutions.

Among the most significant result there is, of course, the recovery of the Strong Equivalence Principle (SEP). In fact, despite that these two alternative theories of gravity are built on different foundation principles with respect to GR, it is necessary that they both recover SEP in order to be considered consistent and to demonstrate their empirical equivalence[1] with GR. However, we must note a non negligible detail: in GR, SEP is a fundamental assumption of the theory, hence postulated *at priori*, while in TEGR and STEGR it is recovered *at posteriori* (see Fig.(6.1) for a representation). Thus, it means that, since in the Geometric Trinity of Gravity it does not seem to be always a fundamental statement, at some level of these formulations it could be not valid anymore [8].

In order to comprehend better this argument, let us think backwards: for instance, if the SEP is a necessary fundamental principle also for TEGR and STEGR, then it is a meta-empirical evidence in favor of what we called the FEP, that is the Equivalence Principle as Fundamental. But, as we have seen, this is not the case. Thus, our confidence in the fundamentality of the SEP decreases. A question should now naturally arise:

[1]Empirical equivalence is a philosophical concept describing two distinct scientific theories that make identical predictions about observable phenomena, meaning no amount of empirical evidence could distinguish between them.

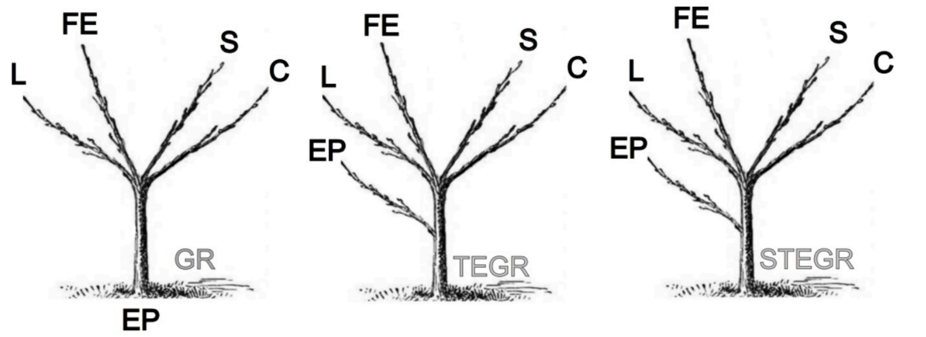



Figure 6.1: The figure shows how the three theories differ with respect to the EP (the SEP, in particular). In GR, EP is at the foundation of the theory and without it the other predictions are not possible; without the root, the branches would not exist. In TEGR and STEGR, on the contrary, the EP is not at the foundation, but it is a lateral branch, recoverable through the general covariance principle and via the coincident gauge, respectively. L: Lagrangians, F E: field equations, S: solutions of the F E, C: cosmological applications. Figure credits [8].

*Could the different foundational role of SEP in the three theories lead to differences in empirical content?*

This discussion is remarkably important at a experimental level as well, since it implies that it is not sufficient to have dinamically equivalent theories at a geometric level to ensure their complete equivalence, because if the fundamental features are different, the experimental protocols needed to test those foundations can also be different. This is due to the fact that if the SEP is not a fundamental feature of reality (FEP), it must be an emergent one (EMEP, EMergent Equivalence Principle). And if it is emergent, it is possible that, at some level, it is not valid.

## 6.5 Open problems and experimental Equivalence Principle violation detection

Before proceeding further, we would like to clarify the main physical issues related to the SEP and how possible violations of it are detected experimentally. Tests of the WEP are those that verify the equivalence of gravitational mass and inertial mass. An obvious test is dropping different objects and verifying that they land at the same time. On the other hand, the SEP can be tested by examining orbital variations in massive systems such as the Sun-Earth-Moon system, detecting possible changes in the gravitational constant $G$ due to nearby gravitational sources or relative motion, or by investigating whether Newton gravitational constant has varied over the history of the universe [42].

To begin with, it is important to note that these tests are extremely complex and require many years of work. Moreover, they must also investigate the possibility of SEP violations occurring within the quantum regime. Indeed, it remains uncertain whether the WEP, and consequently the SEP, hold at the quantum level. At present, we only assume its validity, although it is still unclear whether the WEP can be meaningfully extended to quantum systems [43, 44]. At the quantum level, particles behave as wave packets, making it difficult to give a clear meaning to the concepts of universality of free fall or the equivalence between gravitational and inertial mass (see [45] for procedure details). Upcoming experiments involving free falling quantum systems could provide a direct way to test and explore these ideas [46, 47, 44].

Experimentally, the WEP violation is quantified by the Eötvös parameter $\eta$. It is defined as the ratio [48]

$$\eta_{A,B} = 2\frac{a_A - a_B}{a_A + a_B} \simeq \left(\frac{m_g}{m_i}\right)_A - \left(\frac{m_g}{m_i}\right)_B, \tag{6.5.1}$$

where $a_A$ and $a_B$ are the acceleration of the two free-falling test bodies $A$ and $B$, and $m_g$ and $m_i$ their gravitational and inertial mass, respectively. At the moment, the highest accuracy has been reached by the MICROSCOPE space mission in 2022 [49, 50], with a free-fall experiment performed with macroscopic classical masses, with a Eötvös parameter $\eta = 10^{-15}$, by using two masses of different compositions (titanium and platinum alloys) on a quasi-circular trajectory around the Earth. This is achieved by comparing the accelerations inferred from the forces required to keep the two masses on exactly the same orbit. Any statistically significant difference between the measured accelerations, appearing at a specific frequency, would indicate either a violation of the Weak Equivalence Principle or the presence of a small additional force beyond gravity.
Nonetheless, atomic sensor methods with enhanced performarce for WEP test in space has been proposed by the STE-QUEST (Space-Time Explorer and QUantum Equivalence Space Test) mission and aim to reach a precision of $\sim 10^{-17}$.
Another work in progress experimental field investigates the WEP with antimatter, which makes use of an interferometric configuration to measure the effect of gravity on positronium and antihydrogen [51, 52]. We report in Fig.(6.2) the different tests of WEP from 1960 until today, and the predictions for the future.

Now, if these experiments leads to a violation of some form of the EP, then it means that

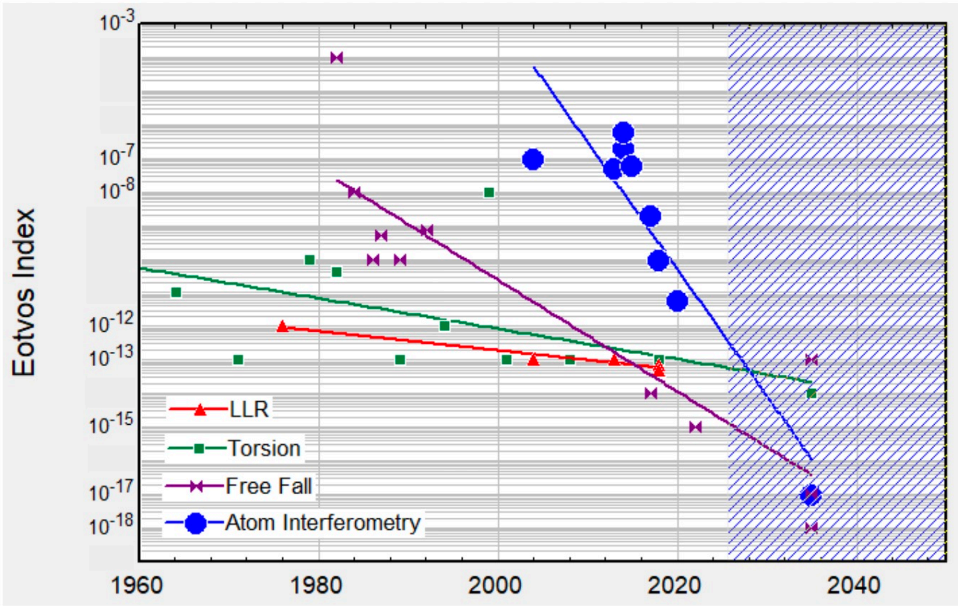



FIGURE 6.2: The most relevant WEP tests performed from 1960. The shaded area contains the prospects of future missions. See [53, 54, 55, 56] for data. Figure credits [8].

GR is not valid at that level anymore. The situation is different for TEGR and STEGR. In fact, since they do not require the EP a priori, they would remain viable theories.

Although this argument makes sense for us, other authors claimed that since GR, TEGR and STEGR are dinamically equivalent, no empirical differences between them should exist. Hence, a SEP violation would falsify all of them. Moreover, SEP could be locally retrieved with a proper gauge choice, meaning that there could be points in the spacetime where SEP does not hold, but this does not falsify the theories, since in these frameworks it is not a fundamental principle. The SEP then would become an emergent principle (EMEP), recoverable only at limited levels, with constraints highlighted by experiments.

It should also be noted that the continued confirmation of the SEP at all levels raises additional conceptual issues. If, after significant experimental progress, no violation is detected, at what level of accuracy should we consider it to be satisfied, $10^{-19}$?$10^{-20}$?$10^{-50}$?

## 6.6 Theoretical argument for identifying a possible Equivalence Principle violation

The Geometric Trinity indicates that there exist consistent formulations of gravity that do not require the EP as a fundamental postulate. As discussed in the previuos section, this could be attributed to the fact that the EP, and then the SEP, are Emergent principles. From a theoretical point of view, considering connections and parallel transport, the Einstein Equivalence Principle (EEP) is closely tied to the Principle of General Covariance and the existence of autoparallel curves. These geometric notions can be reinterpreted through the Noether theorems, which link conserved quantities to internal and spacetime symmetries. Within this framework, it can be shown that the EP, particularly the EEP and the SEP, emerges as a Noether symmetry (see [57] for further details).

Keeping in mind this argument, which includes the use of geodesics and autoparallel curves, we can naturally ask ourselves the following question:

*Is the equivalence of the actions and the field equations sufficient to state that GR, TEGR and STEGR are dynamically equivalent?*

To answer this question, we must understand where, at a geometrical level, we can address a violation of the Equivalence Principle. To do this, we first refer to one of the most important paper in the subject, "*The geometry of free-fall and light propagation*" [58]. In this article, it has been shown how it is possible to derive the Lorentzian geometry, that underlies the spacetime of GR, from a compatible conformal and projective structures on a 4-dimensional manifold (see Appendix (B) for details). EPS introduce the concept of *universality* of geodesics structure, which should be understood in a projective way: the free fall of bodies only determines the class of non-parametrized geodesics, not a complete affine connection nor a unique metric. Two theories could share the same free fall trajectories even if they have two different affine connections, and in general a map which identifies an affine structure with another one does not exist if they do not belong to the same projective class. This identification is possible only if the theory is totally metric, i.e. when the connection is the Levi-Civita one, and the geodesics structure (along with the conformal one fixed by the light propagation) determines the metric up to conformal factors. In this scenario, the geodesics identification leads to the metrics identification.

On the other hand, in non-metric theories or metric-affine ones this does not hold. For this reason, in purely metric theories the Strong Equivalence Principle is automatically satisfied, while in non-metric theories it could fail even though the free fall is universal.

Now the breaking point: what if the theory is non-metric? We can find the answer in the article "*Motion of test particle in spacetimes with torsion and nonmetricity*" of Iosifidis and Hehl [9], which we will briefly discuss below.

### 6.6.1 Test particle equations of motion in Metric-Affine Gravity

As we have seen in Sec.(3.2), MAG theories can be treated *à la Palatini*. Then, it is possible to identify the sources of the theory with the variations of $\mathcal{L}_m$ as hypermomentum $\Delta_\alpha{}^{\mu\nu}$, (metrical) energy-momentum tensor $\mathfrak{T}_{\mu\nu}$ and canonical energy-momentum tensor $\Sigma^\mu{}_\nu$.

Moreover, it is important to note that in the STEGR scenario (see Sec.(5.3)), we set to zero the hypermomentum to recover the connection field equations (5.3.37). We now want to keep this quantity non-zero, since its presence, together with the energy momentum-tensor, exite the spacetime curvature, torsion and non-metricity.

In order to obtain the equation of motion of a test particle, the authors proceeded in three steps:

1. retrieving the MAG conservation laws by taking in account the invariance of matter action under diffeomorphism and the general linear group $GL(4,\mathbb{R})$ [9, 59]. These are

$$\frac{1}{\sqrt{-g}}(\nabla_\mu - 2T_\mu)(\sqrt{-g}\Sigma^\mu{}_\alpha) = -2T_{\alpha\mu\nu}\Sigma^{\mu\nu} + \frac{1}{2}R_{\lambda\mu\nu\alpha}\Delta^{\lambda\mu\nu} - \frac{1}{2}Q_{\alpha\mu\nu}\mathfrak{T}^{\mu\nu}; \tag{6.6.1}$$

$$\frac{1}{2\sqrt{-g}}(\nabla_\nu - 2T_\nu)(\sqrt{-g}\Delta_\lambda{}^{\mu\nu}) = \Sigma^\mu{}_\nu - \mathfrak{T}^\mu{}_\nu. \tag{6.6.2}$$

2. Assuming the form of $\Sigma^\mu{}_\nu$ and $\Delta_\alpha{}^{\mu\nu}$ to be convective, i.e. for a particle with intrinsic properties, the momentum and hypermomentum density are no longer proportional to their transport velocity [9]. For what concerns the metric energy-momentum tensor $\mathfrak{T}^{\mu\nu}$, it is instead assumed to be that of a structurless point particle.

3. Approximating the test particle as point particle, by means of a Dirac delta function.

We obtain

$$\Sigma^\mu{}_\nu = \frac{1}{\sqrt{-g}}u^\mu p_\nu \frac{d\tau}{dt}\delta^{(3)}(\vec{x} - \vec{x}_p(t)), \tag{6.6.3}$$

$$\Delta_\lambda{}^{\mu\nu} = \frac{1}{\sqrt{-g}}u^\nu h_\lambda{}^\mu \frac{d\tau}{dt}\delta^{(3)}(\vec{x} - \vec{x}_p(t)), \tag{6.6.4}$$

$$\mathfrak{T}^{\mu\nu} = \frac{1}{\sqrt{-g}}m u^\mu u^\nu \frac{d\tau}{dt}\delta^{(3)}(\vec{x} - \vec{x}_p(t)), \tag{6.6.5}$$

where $m$ is the mass of the particle and $\vec{x}_p(t)$ its postision in $3d$-space.
From the integral of densities of $\Sigma^\mu{}_\nu$ and $\Delta_\lambda{}^{\mu\nu}$ it follows that

$$P_\nu = \int d^3x\sqrt{\gamma}p_\nu\delta^{(3)}(\vec{x} - \vec{x}_p(t)), \tag{6.6.6}$$

$$H_\lambda{}^\mu = \int d^3x\sqrt{\gamma}h_\lambda{}^\mu\delta^{(3)}(\vec{x} - \vec{x}_p(t)), \tag{6.6.7}$$

which are the total momentum and total hypermomentum, respectively.
Integrating the conservation laws and using the previous definitions, it is possible to retrieve the trajectory of a particle fully charged under hypermomentum, i.e. dilation, spin and shear together:

$$\boxed{m\left(\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2} + \mathring{\Gamma}^\nu{}_{\alpha\beta}u^\alpha u^\beta\right) = \frac{1}{2}H^{\alpha\beta}u^\gamma R_{\alpha\beta\gamma}{}^\nu - g^{\lambda\mu}u^\alpha\nabla_\alpha\xi_\lambda}, \tag{6.6.8}$$

where $\xi_\lambda$ is defined through the decomposition of the total momentum as[2]

$$P_\lambda = mu_\lambda + \xi_\lambda, \quad \text{with} \quad \xi_\lambda = -\frac{1}{2}u_\mu u^\nu\nabla_\nu H_\lambda{}^\mu. \tag{6.6.9}$$

We can now analyze eq.(6.6.8) in our field of interest, i.e. in GR, TEGR and STEGR. The first difference is that GR does not support the presence of the hypermomentum, since it is due to the effect of a non-Reamannian geometry. Hence in a metric theory as GR, the hypermomentum vanishes by construction. In this case eq.(6.6.8) becomes

$$\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2} + \mathring{\Gamma}^\nu{}_{\alpha\beta}u^\alpha u^\beta = 0, \tag{6.6.10}$$

which is the usual form of the geodesic equation.
In the TEGR and STEGR scenario, hypermomentum does not vanish in general. Nonetheless, the Riemann tensor is set to zero and we remain with

$$\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2} + \mathring{\Gamma}^\nu{}_{\alpha\beta}u^\alpha u^\beta = -\frac{1}{m}g^{\lambda\mu}u^\alpha\nabla_\alpha\xi_\lambda. \tag{6.6.11}$$

If we want to consider these three theories a Trinity of gravity, we should also require the equivalence of the equation of motion of a test particle, which are always to be considered

[2] In order to describe completely the system, we also need the evolution equation for hypermomentum (see [9] for further details).

when discussing a theory of gravity. This is satisfied only if

$$g^{\lambda\mu}u^{\alpha}\nabla_{\alpha}\xi_{\lambda}=0 \implies g^{\lambda\mu}u^{\alpha}\nabla_{\alpha}(u_{\mu}u^{\nu}\nabla_{\nu}H_{\lambda}{}^{\mu})=0, \tag{6.6.12}$$

which imposes on the hypermomentum very strict constraints. Condition (6.6.12) is identically satisfied in TEGR and STEGR and follows from the connection field equation, that in the case of a non-vanishing hypermomentum assumes the form [9]

$$(\nabla_{\nu}-2T_{\nu})(\sqrt{-g}\Delta_{\lambda}{}^{\mu\nu})=0. \tag{6.6.13}$$

However, a problem arises from this equation, since it follows from eq.(6.6.2) that

$$\Sigma^{\mu}{}_{\nu}=\mathfrak{T}^{\mu}{}_{\nu}, \tag{6.6.14}$$

which imposes the equivalence between the canonical and the metrical energy-momentum tensors, meaning that they both must be symmetric. This is not true in general. For example, fermionic fields have a non-symmetric canonical energy-momentum tensor, thus imposing the condition (6.6.13) means that we are excluding the presence of standard spinors.

As a result, we conclude that the Trinity of Gravity manifests a complete[3] dynamics equivalence only in cases where the hypermomentum is either severely restricted or entirely absent. Let us take a step back and align our thinking.
At field equations level, GR, TEGR and STEGR are dynamically equivalent, but their foundations are completely different. Now we have all the ingredients to answer the following question:

*When we consider two distinguishable particles, are these three theories still dynamically equivalent?*

In GR, we already know that the free fall of these particles is the same even if they have different spins. In TEGR, since there is torsion, two particles with different spins freely fall in different manner, because the geodesics are not the same. Hence, the equivalence is dependent on the type of matter we are taking in consideration: pointlike particles without internal degrees of freedom (zero hypermomentum) show the same free fall and the same dynamics. Instead, particles with internal degrees of freedom require more attention.
In the next section, a detailed and final discussion will be presented on coupling with matter and on the issues that arise depending on the nature of the particle under consideration.

[3] It means at field equations level and at equations of motion level.

## 6.7 Matter coupling

In generalised geometry frameworks, as Metric-Affine theories, there could arise ambiguity when considering the coupling between matter and spacetime geometry [60]. In this section, we will see that matter fields that share the same dynamics in the absence of gravity, can instead be differentiated by their interaction with it.

Let us recall the method we use in General Relativity to switch on the gravitational interaction: the *minimal coupling principle* (MPC). It consists in making the actions $S$ of the standard model fields $\phi$ locally invariant by promoting the metric of the inertial frame (flat) to the dynamical spacetime metric (curved) and replacing the partial derivative with the covariant ones. Namely,

$$S(\eta, \phi, \partial\phi) \longrightarrow S(g, \phi, \nabla\phi). \tag{6.7.1}$$

MCP is considered as the appropriate prescription to couple gauge fields to a matter sector that is charged under the corresponding group.
As example, we can recall Chap.(1.10): in the case of Electrodynamics we have the coupling $\partial_\mu \longrightarrow D_\mu = \partial_\mu + ieA_\mu$, because the photon field is charged under $U(1)$, meanining that it tranforms non-trivially under this symmetry group. Moreover, for a Dirac spinor we see that $\partial_\mu \longrightarrow \mathcal{D}_\mu = \partial_\mu - \frac{i}{2}\omega^{ab}{}_\mu J_{ab}$, because the spinor is charged under the local Lorentz group.
If we consider a point-like particle with standard action $S = m\int \mathrm{d}\tau$, we are considering a purely metric quantity. The equations of motion are geodesics, hence they depend only on the Levi-Civita connection (geodesics coincide with autoparallel curves). This is also a consequence of the MCP for bosonic and fermionic fields in spacetimes equipped with just the metric connnection.

In generalised spacetimes, the MCP does not necessarily lead to standard matter coupling, especially if there is torsion. Hence, the violation of the MCP will thus serve as an indicator to determine whether a non-metric theory, when coupled to gravity, generates additional contributions in the equations of motion or within the field dynamics.

### 6.7.1 Matter coupling in Metric and Symmetric Teleparallel Gravity

Traditionally, we distinguish particles by the transformation properties of their fields under the Lorentz group. We call *boson* a field that belongs to some tensor product of vector representation, and *fermion* a field that belongs to the universal double cover of $SO(3,1)$.

These properties can be extended in a more general framework, in which gravity is turned on, but as we will briefly discuss now, boson fields couple to gravity easier than fermions, as the former only couple to the metric, while the latter also couple to the connection.

#### 6.7.1.1 Bosonic fields

The reason behind the previuos statement is that bosons are described by tensorial representations, while spinors need spinoral fields.
The starting point to introduce gravity is a flat spacetime with a Lorentzian structure, where bosonic fields are described by Lorentz tensors, indeed. Switching on gravity, Lorentz tensors become $GL(4,\mathbb{R})$-tensors and this leads to the possibility of mapping the connection on the $SO(3,1)$ Lorentz bundle to an affine connection on the $GL(4,\mathbb{R})$-bundle.

| **Gravity off** | **Gravity on** |
|---|---|
| Lorentz flat spacetime | (non)-Riemannian geometry |
| Lorentz tensors | $GL(4,\mathbb{R})$-tensors |
| $SO(3,1)$ connection on Lorentz bundle | Affine connection on the $GL(4,\mathbb{R})$-bundle |

Hence, there exists an isomorphism between the tensorial representations of $SO(3,1)$ and $GL(4,\mathbb{R})$. This fact, simplifies the introduction of the coupling to gravity with the covariant derivative obeying the MCP.

#### TEGR scenario

In order to investigate the behaviour of fields in these theories, we introduce the following Lemma (see [61] for details and proofs):

**Lemma 6.7.1.** Let $\mathbf{\Lambda}$ be a gauge field and D the gauge-covariant exterior derivative. Then,

1. the canonical field strength $\mathbf{F}$ is $\mathbf{F} = \mathrm{D}\mathbf{\Lambda}$ iff the connection is torsion-less.
2. If 1. is satisfied, then the Bianchi identity can be written as $\mathrm{D}\mathbf{F} = 0$.

This Lemma tells us that metric TG presents problems even just for a bosonic field if one adops the MCP: even though scalar fields do not couple with torsion, MCP formalism is based on geometric rules as $\mathrm{D}^2 = \mathbf{F}$ and $\mathrm{D}\mathbf{F} = 0$, which hold only if the connection is symmetric (no torsion). If there is torsion, as in metric TG, MCP leads us to write operators that do not respect these identities, hence all the geometric construction becomes potentially non-coherent and $\mathrm{D}\mathbf{F} \neq 0$ in general.

Let us take as example the photon field. In general, we describe it as $\mathbf{A} = \mathbf{A}_\mu \mathrm{d}x^\mu$, with field strength $\mathbf{F} = \mathrm{d}\mathbf{A}$ such that $\mathrm{d}\mathbf{F} = 0$ (photon gauge invariance). If we apply the MCP in a spacetime with torsion, then $\mathrm{d} \longrightarrow \mathrm{D}$ and $\mathbf{F} = \mathrm{D}\mathbf{A}$. This time, from Lemma(6.7.1) follows that $\mathrm{D}^2 \neq 0$ and $\mathrm{D}\mathbf{F} \neq 0$, and especially $\mathrm{D}\mathbf{A} = \mathrm{d}\mathbf{A} + \underbrace{\text{(torsion terms)}}_{\text{non-minimal coupling}}$, namely

$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} \longrightarrow 2\nabla_{[\mu}A_{\nu]} = 2\partial_{[\mu}A_{\nu]} + T^\alpha{}_{\mu\nu}A_\alpha.$$

Thus, the torsion forces the photon to lose its $U(1)$ standard invariance. This is not physically acceptable, since electromagnetic gauge invariance is extremely well tested experimentally. Consequently, in this case the minimal coupling principle cannot be regarded as universal: it may serve as a guiding prescription, but it cannot be applied blindly to all gauge theories. In particular, the electromagnetic field does not couple directly to spacetime torsion. Therefore, even in the presence of torsion, the field strength is defined as

$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}.$$

**STEGR scenario**

In STEGR and GR, we avoid this problem since the torsion vanishes and $\mathbf{F} = \mathrm{D}\mathbf{A}$ as in a Riemann theory.

#### 6.7.1.2 Fermionic fields

On the other hand, for fermions, mapping the $SO(3,1)$ connection on the Lorentz bundle to the affine connection on the $GL(4,\mathbb{R})$-bundle is not straightforward. This is because one must first construct the universal double cover of the Lorentz group, and in general, a direct translation to $GL(4,\mathbb{R})$ is not possible. Hence, in this case does not exist an isomorphism between spinorial representations $SL(2,\mathbb{C})$ and $GL(4,\mathbb{R})$, consequently it is not possible to define the corresponding connection.
Recall in fact that already in the standard formulation of GR it is required the introduction of a spin connection in order to deal with fermions.

**TEGR scenario**

Coupling Dirac spinors in Teleparallel Gravity requires special care because spinors transform under the local Lorentz group. Besides the mathematical argument presented before,

it is important to remark that the rise of inconsistencies is due to the fact that TG is constructed as a gauge theory of the translational group of spacetime. The energy-momentum tensor is the corresponding dynamical current for the generators of translations, while spin current corresponds to generators of the Lorentz group, which is incompatible with the gauge approach based on the translational group [62].
In the traditional "pure-tetrad" formulation of Teleparallel Gravity, the spin connection is artificially set to zero, ${\omega^{ab}}_{\mu} = 0$, in all frames. Since the spin connection is then no longer treated as an independent variable, the minimal coupling prescription becomes inconsistent: the canonical energy-momentum tensor of the Dirac field is non-symmetric, while the metrical energy-momentum tensor is symmetric. As a consequence, the resulting field equations impose spurious constraints on the spinor current, constraints that have no analogue in GR [20, 60]. This problem originates from the fact that setting ${\omega^{ab}}_{\mu} = 0$ breaks local Lorentz invariance and eliminates the inertial part of the spin connection that is required for a consistent spinor coupling (recall the Fock-Ivanenko derivative).

It has been shown in [62] that in order to make spinning matter couple to gravity consistently, we can act in two different ways:

- introduce a different coupling rule such that the canonical energy-momentum tensor becomes symmetric.
  In this case, it is possible to retrieve the correct interaction with the Dirac field if the coupling is well-chosen [63]: the trick lies on assuming that the coupling Lagrangian of the spinor field does not contain the Weitzenböck connection, but only the Levi-Civita part (formally obtaining a description equivalent to the GR).
- Change the dynamical scheme and make it non-symmetric.
  In fact, by including the spin together with the energy-momentum current, as dynamical sources of equal right for the gravitational field, we end up in constructing a gauge theory based on the Poincarè group. Hence, this time there are both the generators of translations and of the Lorentz group, which are related to the canonical energy-momentum tensor and spin, respectively.
  This extension of the dynamical contents yields an extension of the spacetime geometry to the Riemann-Cartan, with non-trivial curvature and torsion.

Both approaches make the inconsistencies disappear and the TG Dirac Lagrangian is exactly equivalent to the GR Dirac Lagrangian[4].

[4]It is done making use of the identity (4.3.19), ${\overset{\bullet}{\omega}^{c}}_{b\mu} - {\hat{K}^{c}}_{b\mu} = {\overset{\circ}{\omega}^{c}}_{b\mu}$.

### STEGR scenario

In STEGR, we can stick with the minimal coupling prescription without making changes. In the coincident gauge, the derivatives $\partial_\mu$ can be understood as $\nabla_\mu$, even when gravity is turned on. Spinors now couple only with the Levi-Civita part of the connection. This is due to the fact that in STEGR the torsion vanishes, and to the Hermitianity of the Dirac action [64].

# Discussion and Outlooks

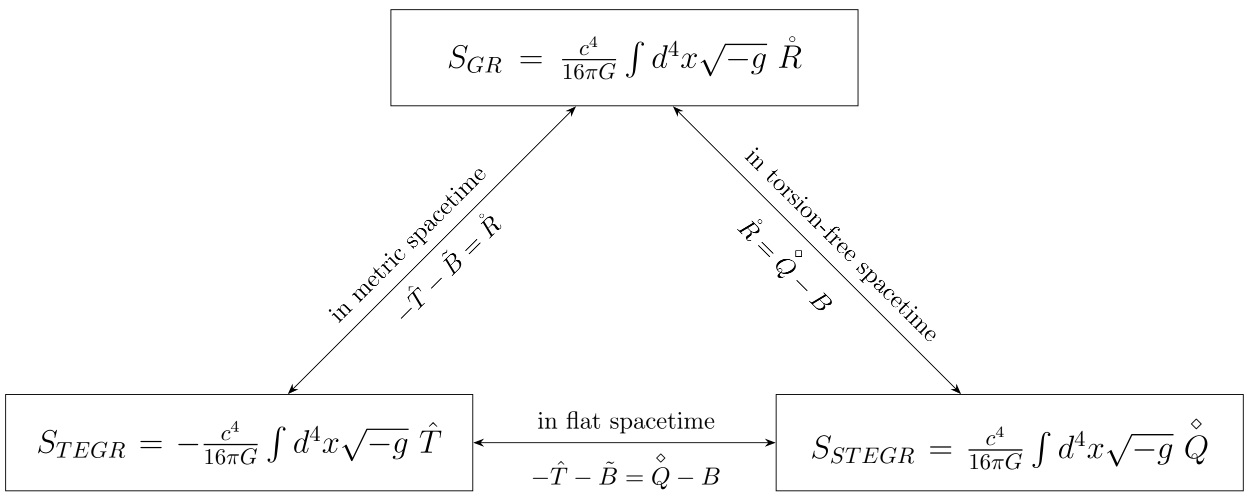
The aim of this work was to investigate alternative approaches in order to geometrise the gravitational interaction, leading to the Geometric Trinity of Gravity.
First of all, since this analysis requires a solid understanding of advanced differential geometry tools, Chapter 1 retraces all the steps to stablish the fundamentals of our framework. Consequently, in Chapter 2 we briefly presented the usual description of General Relativity, showing the principles on which it is founded, with a particular accent on the different versions of the Equivalence Principle and their consequences. Then, we have introduced the Metric-Affine Theories of Gravity in Chapter 3 and proposed a gauge-based formulation of Gravity in Chapter 4, which goes under the name of Teleparallel Equivalent of General Relatity (TEGR), formulated in terms of torsion $\hat{T}$ and relying on tetrads and a flat spin connection: the former generalizes the choice of basis for the tangent bundle from a coordinate basis, while the latter is a $\mathfrak{so}(3,1)$-valued 1-form representing the gauge field generated by local Lorentz transformations.

TEGR is a true gauge theory by definition and, to be precise, it is for the group of spacetime translations. We want to stress that the Teleparallel gauge structure lais on spacetime itself and this is made possible by the capability to promote the translational gloabal symmetry to local symmetry and the introduction of a gauge field associated to the translational group. We defined the fundamental connection of Teleparallel Gravity, namely the Weitzenböck connection, which is flat and non-symmetric. Thanks to this and the Weitzenböck gauge, it is possible to formulate the Teleparallel gravitational coupling prescription and state that it is the same as GR's. Moreover, we comprehend one of the theoretical strengths of this theory: unlike GR, TEGR allows to separate gravitational effects from the inertial ones, and consequently to restore the Strong Equivalence Principle (SEP).

We derived the expression for the Ricci scalar within this framework using the Riemann tensor. In this context, the Ricci scalar $\mathring{R}$ and the torsion scalar $\hat{T}$ differ by a boundary term $\tilde{B}$, which arises from the specific choice of connection (the Weitzenböck gauge). As a consequence, the GR Lagrangian coincides with the TEGR Lagrangian, with a particular choice of the torsion scalar, up to the term $\tilde{B}$.

We applied a similar procedure in Chapter 5 in the case of the Symmetric Teleparallel

Gravity, which gives rise to the Symmetric Teleparallel Equivalent of General Relativity (STEGR), based on non-metricity $\overset{\diamond}{Q}$ and constructed from the metric together with a flat and torsionless affine connection. Also in this case, we have derived the Ricci scalar in terms of the non-metricity tensor $\overset{\diamond}{Q}$ and a boundary term $B$ and showed that the GR Lagrangian and STEGR Lagrangian only differ by that boundary term, this time using the coincident gauge.
We have emphasized that, unlike TEGR, the STEGR is not a guage theory by definition but it can be interpreted as one by admitting a gauge-theoretic structure when the Strong Equivalence Principle itself is viewed as a gauge symmetry.
GR, TEGR and STEGR turn out to be dynamically equivalent: their Lagrangians differ only by boundary terms and they lead to the same form of field equations and solutions.
We want to stress that the equivalence holds because the boundary terms do not contribute to the dynamics since the spacetime is assumed to be asymptotically flat at infinity, hence they vanish. These three theories then constitute the so-called Geometric Trinity of Gravity, and we can summarize their main differences and their equivalence with the following diagram:




In Chapter 6, we have finally presented the open issues and tensions of the Trinity of Gravity. First of all, we have seen that these three theories are dinamically equivalent, while being completely different in their foundations. In fact, we have seen that in GR, the SEP is a fundamental assumption of the theory, hence postulated *at priori*, while in TEGR and STEGR it is recovered *at posteriori*. Thus, it means that, since in the Geometric Trinity of Gravity it does not seem to be always a fundamental statement, at some level of these formulations it could be not valid anymore. After that, we have addressed a problem strictly related to the SEP: the free fall of test particles in these theories. In all the metric theories,

as GR, there is the universality of free fall, meaning that different test particles, with or without internal structure, undergo the same free fall. Moreover, geodesics do coincide with autoparallel curves.
Things get more delicate when dealing with non-metric theories. Here, geodesics do not coincide with autoparallel curves due to the presence of additional terms of the generalised connection. In these cases, the internal structure of test particles, described by the hypermomentum, makes different equations of motion depending on the type of particle we are considering, unless it is either severely restricted or entirely absent. Hence, we can conclude that the equivalence in the Trinity depends on the type of matter we are dealing with: point-like particles without internal degrees of freedom (zero hypermomentum) show the same free fall and the same dynamics in all the theories, but even just the presence of spin forces to admit different free fall.

In conclusion, we have discussed the ambiguities arising in the coupling between matter and spacetime geometry within Metric-Affine theories. In generalized spacetimes, the Minimal Coupling Prescription (MCP) does not necessarily reproduce the standard matter coupling, particularly in the presence of torsion. For this reason, violations of the MCP can be used as an indicator to determine whether a given non-metric theory, once coupled to gravity, generates additional contributions to the equations of motion or modifies the field dynamics.

Concerning bosonic fields, there exists an equivalence between tensorial representations of the Lorentz group $SO(3,1)$ and those of $GL(4,\mathbb{R})$. This property simplifies the introduction of the gravitational coupling, allowing the use of a covariant derivative that formally obeys the MCP.

The situation is significantly different for fermionic fields. Since spinors transform under the spin group $SL(2,\mathbb{C})$, which does not admit a representation within $GL(4,\mathbb{R})$, it is not possible to define a corresponding affine connection acting on spinors. As a consequence, fermion coupling requires the introduction of a spin connection associated with local Lorentz invariance. To better illustrate these issues, we compared the TEGR and the STEGR.

In TEGR, the application of the MCP is more delicate for both bosons and fermions. For bosonic fields, the MCP leads to operators that fail to satisfy the expected identities, potentially rendering the geometric construction inconsistent. For this reason, we do not apply the MCP, and we define bosonic gauge fields independently of the Teleparallel connection, so that their field strengths retain their standard gauge-invariant form.
For Dirac spinors, additional complications arise because spinors transform under the local

Lorentz group, which is not naturally compatible with the gauge formulation based on spacetime translations. A possible resolution consists in restricting the coupling to the Levi-Civita part of the connection. In this case, the resulting Dirac Lagrangian becomes dynamically equivalent to that of General Relativity, thereby restoring consistency.

In STEGR, the geometric setting considerably simplifies the coupling to matter fields. In fact, the vanishing property of the torsion allows the straightforward introduction of a covariant derivative, using the coincident gauge, that consistently implements the MCP, without generating spurious geometric contributions in the field equations.

Fermionic fields also admit a more transparent treatment in STEGR compared to torsion-based theories. Since torsion vanishes identically, the ambiguities typically associated with the spin-torsion coupling are absent. The coupling of Dirac spinors can be consistently defined by introducing a spin connection associated with local Lorentz invariance, which can be chosen to be purely inertial. As a result, the fermionic sector does not acquire additional non-metric contributions beyond those already present in General Relativity and the MCP is preserved.

The findings of this thesis suggest several possible directions for future work.
First, it is natural to ask whether TEGR and STEGR are the only theories dynamically equivalent to GR, or if alternative representations of Gravity equivalent to GR may exist. Finding other dynamically equivalent theories would help in exploring different way to geometrise gravitational interaction, evaluate other possible relevant observables and determine whether the SEP is a fundamental or an emergent principle. Thus, an important direction for future research lies in developing experimental approaches that go beyond traditional tests of the EP, particularly at the quantum level and in currently unexplored regimes, which may allow one to discriminate among the equivalent theories.
This leads to another interesting point: the construction of observational apparatus capable of discriminating between the different sets of observables associated with equivalent descriptions of Gravity. From this perspective, experimental evidence could help clarify why these theories are dynamically equivalent and identify the transformation laws that relate them.
Finally, further investigation of the STEGR approach is warranted, since it remains less developed than TEGR. A general tetrad formulation is particularly important for understanding its physical implications, especially from a gauge-theoretic perspective.

# Appendices

# Appendix A

# Computation of TEGR field equations

In this section, we will show the complete derivation of the TEGR field equations, presented in Sec.(4.3.4).
Let us start by recalling the field equations we have to compute and the TEGR Lagrangian:

$$\frac{\partial \mathcal{L}_{TEGR}}{\partial e^a{}_\rho} - \partial_\sigma \frac{\partial \mathcal{L}_{TEGR}}{\partial(\partial_\sigma e^a{}_\rho)} = 0, \tag{A.0.1}$$

where

$$\mathcal{L}_{TEGR} = \frac{c^4 e}{16\pi G}\left[\frac{1}{4}\left(\hat{T}^{\alpha\mu\nu}\hat{T}_{\alpha\mu\nu} + 2\hat{T}_{\alpha\mu\nu}\hat{T}^{\mu\alpha\nu}\right) - \hat{T}^\alpha \hat{T}_\alpha\right], \tag{A.0.2}$$

with

$$\hat{T}^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \overset{\bullet}{\omega}{}^a{}_{b\mu} e^b{}_\nu - \overset{\bullet}{\omega}{}^a{}_{b\nu} e^b{}_\mu \tag{A.0.3}$$

and $\hat{T}^\rho{}_{\mu\nu} = e_a{}^\rho \hat{T}^a{}_{\mu\nu}$.
To find the field equations, we need to compute the derivative of the Lagrangian with respect to the tetrad field. We start from

$$\frac{\partial \mathcal{L}_{TEGR}}{\partial(\partial_\sigma e^a{}_\rho)} = \frac{c^4 e}{16\pi G}\frac{\partial}{\partial(\partial_\sigma e^a{}_\rho)}\left[\frac{1}{4}\hat{T}^{\alpha\mu\nu}\hat{T}_{\alpha\mu\nu} + \frac{1}{2}\hat{T}_{\alpha\mu\nu}\hat{T}^{\mu\alpha\nu} - \hat{T}^\alpha \hat{T}_\alpha\right]. \tag{A.0.4}$$

The first term is

$$\begin{aligned}
\frac{\partial}{\partial(\partial_\sigma e^a{}_\rho)}\left[\frac{1}{4}\hat{T}^{\alpha\mu\nu}\hat{T}_{\alpha\mu\nu}\right] &= \frac{\partial}{\partial(\partial_\sigma e^a{}_\rho)}\left[\frac{1}{4}\overbrace{e_c{}^\alpha e^b{}_\alpha}^{\delta^b{}_c}\hat{T}^c{}_{\mu\nu}\hat{T}_b{}^{\mu\nu}\right]\\
&= \frac{\partial}{\partial(\partial_\sigma e^a{}_\rho)}\left[\frac{1}{4}\big(\hat{T}_b{}^{\mu\nu}\big)^2\right]\\
&= \frac{1}{4}2\hat{T}_b{}^{\mu\nu}\frac{\partial}{\partial(\partial_\sigma e^a{}_\rho)}\hat{T}^b{}_{\mu\nu}\\
&= \frac{1}{2}\hat{T}_b{}^{\mu\nu}\frac{\partial}{\partial(\partial_\sigma e^a{}_\rho)}\left[\partial_\mu e^b{}_\nu - \partial_\nu e^b{}_\mu + \cancel{\overset{\bullet}{\omega}{}^b{}_{c\mu}e^b{}_\nu} - \cancel{\overset{\bullet}{\omega}{}^b{}_{c\nu}e^c{}_\mu}\right]\\
&= \frac{1}{2}\hat{T}_b{}^{\mu\nu}\big(\delta^\sigma{}_\mu\delta^b{}_a\delta^\rho{}_\nu - \delta^\sigma{}_\nu\delta^b{}_a\delta^\rho{}_\mu\big)\\
&= \frac{1}{2}\hat{T}_a{}^{\sigma\rho} - \frac{1}{2}\hat{T}_a{}^{\rho\sigma} = \hat{T}_a{}^{\sigma\rho},
\end{aligned} \tag{A.0.5}$$

using the antisymmetry of $\hat{T}_a{}^{\mu\nu}$.

In the same way, we can compute the second term of (A.0.4).

$$\begin{aligned}
\frac{\partial}{\partial(\partial_\sigma e^a{}_\rho)}\left[\frac{1}{2}\hat{T}_{\alpha\mu\nu}\hat{T}^{\mu\alpha\nu}\right] &= \frac{\partial}{\partial(\partial_\sigma e^a{}_\rho)}\left[\frac{1}{2}e_c{}^\alpha e^b{}_\alpha\hat{T}^c{}_{\mu\nu}\hat{T}^\mu{}_b{}^\nu\right]\\
&= \frac{\partial}{\partial(\partial_\sigma e^a{}_\rho)}\left[\frac{1}{2}\hat{T}^c{}_{\mu\nu}\hat{T}^{\nu\mu}{}_c\right]\\
&= \frac{1}{2}\frac{\partial\hat{T}^c{}_{\mu\nu}}{\partial(\partial_\sigma e^a{}_\rho)}\hat{T}^{\nu\mu}{}_c + \frac{1}{2}\frac{\partial\hat{T}^{\nu\mu}{}_c}{\partial(\partial_\sigma e^a{}_\rho)}\hat{T}^c{}_{\mu\nu}\\
&= \hat{T}^{\rho\sigma}{}_c - \hat{T}^{\sigma\rho}{}_c.
\end{aligned} \tag{A.0.6}$$

The last term of (A.0.4) is:

$$\begin{aligned}
\frac{\partial}{\partial(\partial_\sigma e^a{}_\rho)}\left[-\hat{T}^\alpha\hat{T}_\alpha\right] &= -2\hat{T}^\alpha\frac{\partial\hat{T}_\alpha}{\partial(\partial_\sigma e^a{}_\rho)}\\
&= -2\hat{T}^{\nu\mu}{}_\nu e_b{}^\lambda\frac{\partial\hat{T}^b{}_{\mu\nu}}{\partial(\partial_\sigma e^a{}_\rho)}\\
&= -2\hat{T}^{\nu\mu}{}_\nu e_b{}^\lambda\big(\delta^b{}_a\delta^\sigma{}_\mu\delta^\rho{}_\lambda - \delta^b{}_a\delta^\sigma{}_\lambda\delta^\rho{}_\mu\big)\\
&= -2\hat{T}^{\nu\sigma}{}_\nu e_a{}^\rho + 2\hat{T}^{\nu\rho}{}_\nu e_a{}^\sigma.
\end{aligned} \tag{A.0.7}$$

From these three terms, we can write eq.(A.0.4) and construct the superpotential:

$$\hat{S}_a{}^{\mu\nu} = -\frac{8\pi G}{c^4 e}\frac{\partial \mathcal{L}_{TEGR}}{\partial(\partial_\nu e^a{}_\mu)} = -\frac{8\pi G}{c^4 e}\frac{c^4 e}{16\pi G}\left(\hat{T}_a{}^{\nu\mu} + \hat{T}^{\mu\nu}{}_a - \hat{T}^{\nu\mu}{}_a - 2\hat{T}^{\alpha\nu}{}_\alpha e_a{}^\mu + 2\hat{T}^{\alpha\mu}{}_\alpha e_a{}^\nu\right), \tag{A.0.8}$$

that can be rewritten as

$$\hat{S}_a{}^{\mu\nu} = \hat{K}^{\mu\nu}{}_a - e_a{}^\nu \hat{T}^\mu + e_a{}^\mu \hat{T}^\nu. \tag{A.0.9}$$

Now, we have to deal with the other derivative of the TEGR field equations (A.0.1), i.e

$$\frac{\partial \mathcal{L}_{TEGR}}{\partial e^a{}_\rho} = \frac{c^4}{16\pi G}\frac{\partial}{\partial e^a{}_\rho}\left\{ e\left[\frac{1}{4}\left(\hat{T}^{\alpha\mu\nu}\hat{T}_{\alpha\mu\nu} + 2\hat{T}_{\alpha\mu\nu}\hat{T}^{\mu\alpha\nu}\right) - \hat{T}^\alpha \hat{T}_\alpha\right]\right\}. \tag{A.0.10}$$

Using the property

$$\frac{\partial e}{\partial e^a{}_\rho} = e e_a{}^\rho, \tag{A.0.11}$$

we can rewrite (A.0.10) as

$$\begin{aligned}\frac{\partial \mathcal{L}_{TEGR}}{\partial e^a{}_\rho} = \frac{c^4 e}{16\pi G}\Bigg[&\frac{1}{4}\frac{\partial \hat{T}^c{}_{\mu\nu}}{\partial e^a{}_\rho}\hat{T}_c{}^{\mu\nu} + \frac{1}{4}\hat{T}^c{}_{\mu\nu}\frac{\partial \hat{T}_c{}^{\mu\nu}}{\partial e^a{}_\rho} + \frac{1}{2}\hat{T}^c{}_{\mu\nu}\frac{\partial \hat{T}^{\nu\mu}{}_c}{\partial e^a{}_\rho} + \frac{1}{2}\frac{\partial \hat{T}^c{}_{\mu\nu}}{\partial e^a{}_\rho}\hat{T}^{\nu\mu}{}_c - \hat{T}_{\lambda\mu}{}^\lambda \frac{\partial \hat{T}^{\nu\mu}{}_\nu}{\partial e^a{}_\rho} \\ &- \frac{\partial \hat{T}_{\lambda\mu}{}^\lambda}{\partial e^a{}_\rho}\hat{T}^{\nu\mu}{}_\nu\Bigg] + \underbrace{\frac{c^4}{16\pi G} e e_a{}^\rho\left[\frac{1}{4}\left(\hat{T}^{\alpha\mu\nu}\hat{T}_{\alpha\mu\nu} + 2\hat{T}_{\alpha\mu\nu}\hat{T}^{\mu\alpha\nu}\right) - \hat{T}^\alpha\hat{T}_\alpha\right]}_{e_a{}^\rho \mathcal{L}_{TEGR}}.\end{aligned} \tag{A.0.12}$$

We have to compute the derivatives of (A.0.12). Its first and fourth term can be computed in the same way using the definition of the torsion tensor:

$$\begin{aligned}\frac{\partial \hat{T}^c{}_{\mu\nu}}{\partial e^a{}_\rho} &= \frac{\partial}{\partial e^a{}_\rho}\left[\cancel{\partial_\mu e^c{}_\nu} - \cancel{\partial_\nu e^c{}_\mu} + \overset{\bullet}{\omega}{}^c{}_{b\mu} e^b{}_\nu - \overset{\bullet}{\omega}{}^c{}_{b\nu} e^b{}_\mu\right] \\ &= \overset{\bullet}{\omega}{}^c{}_{b\mu}\delta^b{}_a \delta^\rho{}_\nu - \overset{\bullet}{\omega}{}^c{}_{b\nu}\delta^b{}_a\delta^\rho{}_\mu \\ &= \overset{\bullet}{\omega}{}^c{}_{a\mu}\delta^\rho{}_\nu - \overset{\bullet}{\omega}{}^c{}_{a\nu}\delta^\rho{}_\mu.\end{aligned} \tag{A.0.13}$$

In order to compute the other derivatives of (A.0.12), we need three properties:

$$\frac{\partial e_c{}^\nu}{\partial e^a{}_\rho} = -e_a{}^\nu e_c{}^\rho, \tag{A.0.14}$$

$$\frac{\partial g^{\mu\nu}}{\partial e^c{}_\rho} = -g^{\rho\nu} e_c{}^\mu - g^{\rho\mu} e_c{}^\nu, \tag{A.0.15}$$

$$\frac{\partial g_{\nu\sigma}}{e^a{}_\rho} = \delta^\rho{}_\lambda e_{a\nu} + \delta^\rho{}_\nu e_{a\lambda}. \tag{A.0.16}$$

which follow form the orthogonality of the tedrad and of the metric, respectively. Then, the second term of (A.0.12) is

$$\begin{aligned}
\frac{\partial \hat{T}_c{}^{\mu\nu}}{\partial e^a{}_\rho} &= \frac{\partial}{\partial e^a{}_\rho}\Big[\eta_{cb} g^{\mu\alpha} g^{\nu\beta} \hat{T}^b{}_{\alpha\beta}\Big] \\
&= \eta_{cb}\frac{\partial g^{\mu\alpha}}{\partial e^a{}_\rho} g^{\nu\beta}\hat{T}^b{}_{\alpha\beta} + \eta_{cb}\frac{\partial g^{\nu\beta}}{\partial e^a{}_\rho} g^{\mu\alpha}\hat{T}^b{}_{\alpha\beta} + \eta_{cb} g^{\mu\alpha} g^{\nu\beta}\frac{\partial \hat{T}^b{}_{\alpha\beta}}{\partial e^a{}_\rho} \\
&= \Big[ - g^{\nu\beta} g^{\alpha\rho} e_a{}^\mu - g^{\nu\beta} g^{\mu\rho} e_a{}^\alpha - g^{\mu\alpha} g^{\nu\rho} e_a{}^\beta - g^{\mu\alpha} g^{\beta\rho} e_a{}^\nu \Big] \hat{T}_{c\alpha\beta} \\
&\qquad + \eta_{cb} g^{\mu\alpha} g^{\nu\beta}\Big[\overset{\bullet}{\omega}{}^b{}_{a\alpha}\delta^\rho{}_\beta - \overset{\bullet}{\omega}{}^b{}_{a\beta}\delta^\rho{}_\alpha\Big] \\
&= -e_a{}^\mu \hat{T}_c{}^{\rho\nu} - g^{\mu\rho}\hat{T}_{ca}{}^\nu - g^{\nu\rho}\hat{T}_c{}^\mu{}_a - e_a{}^\nu \hat{T}_c{}^{\mu\rho} \\
&\qquad + \eta_{cb} g^{\mu\alpha} g^{\nu\rho}\overset{\bullet}{\omega}{}^b{}_{a\alpha} - \eta_{cb} g^{\mu\rho} g^{\nu\beta}\overset{\bullet}{\omega}{}^b{}_{a\beta}.
\end{aligned} \tag{A.0.17}$$

The derivative of third term of (A.0.12) is

$$\begin{aligned}
\frac{\partial \hat{T}^{\nu\mu}{}_c}{\partial e^a{}_\rho} &= \frac{\partial}{\partial e^a{}_\rho}\Big[\eta_{cb} e^b{}_\sigma \hat{T}^{\nu\mu\sigma}\Big] \\
&= \eta_{ca}\hat{T}^{\nu\mu\rho} + \eta_{cb} e^b{}_\sigma \frac{\partial}{\partial e^a{}_\rho}\Big(\eta^{bc} e_b{}^\nu \hat{T}_c{}^{\mu\sigma}\Big) \\
&= \eta_{ca}\hat{T}^{\nu\mu\rho} + \eta_{cb} e^b{}_\sigma\Big( - \eta^{bc} e_a{}^\nu e_b{}^\rho \hat{T}_c{}^{\mu\sigma} + \eta^{bc} e_b{}^\nu \frac{\partial \hat{T}_c{}^{\mu\sigma}}{\partial e^a{}_\rho}\Big) \\
&= -e_a{}^\nu \hat{T}^{\rho\mu}{}_c - e_a{}^\mu \hat{T}^{\nu\rho}{}_c - e_c{}^\rho \hat{T}^{\nu\mu}{}_a - g^{\mu\rho}\hat{T}^\nu{}_{ac} + e_b{}^\nu\big(g^{\mu\lambda} g^{\sigma\rho} - g^{\mu\rho} g^{\sigma\lambda}\big)\overset{\bullet}{\omega}{}^b{}_{a\lambda},
\end{aligned} \tag{A.0.18}$$

where we have used the result of (A.0.17) in the last step.
The derivative in the fifth term of (A.0.12) is

$$
\begin{aligned}
\frac{\partial \hat{T}^{\nu\mu}{}_{\nu}}{\partial e^{a}{}_{\rho}} &= \frac{\partial}{\partial e^{a}{}_{\rho}}\left[g_{\nu\sigma}\hat{T}^{\nu\mu\sigma}\right] \\
&= \frac{\partial g_{\nu\sigma}}{\partial e^{a}{}_{\rho}}\hat{T}^{\nu\mu\sigma} + g_{\nu\sigma}\frac{\partial \hat{T}^{\nu\mu\sigma}}{\partial e^{a}{}_{\rho}} \\
&= -\hat{T}^{\rho\mu}{}_{a} - e_{a}{}^{\mu}\hat{T}^{\nu\rho}{}_{\nu} - g^{\mu\rho}\hat{T}^{\nu}{}_{a\nu} + e_{a}{}^{\rho}g^{\mu\nu}\overset{\bullet}{\omega}{}^{b}{}_{a\nu} - e_{b}{}^{\nu}g^{\mu\rho}\overset{\bullet}{\omega}{}^{b}{}_{a\nu}.
\end{aligned} \tag{A.0.19}
$$

Finally, the derivative of the last term of (A.0.12) is

$$
\begin{aligned}
\frac{\partial \hat{T}_{\lambda\mu}{}^{\lambda}}{\partial e^{a}{}_{\rho}} &= \frac{\partial}{\partial e^{a}{}_{\rho}}\left[e_{c}{}^{\lambda}\hat{T}^{c}{}_{\mu\lambda}\right] \\
&= -e_{a}{}^{\lambda}e_{c}{}^{\rho}\hat{T}^{c}{}_{\mu\lambda} + e_{c}{}^{\lambda}\left[\overset{\bullet}{\omega}{}^{c}{}_{a\mu}\delta^{\rho}{}_{\lambda} - \overset{\bullet}{\omega}{}^{c}{}_{a\lambda}\delta^{\rho}{}_{\mu}\right] \\
&= -\hat{T}^{\rho}{}_{\mu a} + e_{c}{}^{\rho}\overset{\bullet}{\omega}{}^{c}{}_{a\mu} - e_{c}{}^{\lambda}\overset{\bullet}{\omega}{}^{c}{}_{a\lambda}\delta^{\rho}{}_{\mu}.
\end{aligned} \tag{A.0.20}
$$

Using all these derivative terms, we can assemble the eq.(A.0.12) and the write the energy-momentum pseudo tensor:

$$
\mathfrak{t}_{a}{}^{\rho} = -\frac{1}{e}\frac{\partial \mathcal{L}_{TEGR}}{\partial e^{a}{}_{\rho}} = \frac{c^4}{8\pi G}e_{a}{}^{\lambda}\hat{S}_{c}{}^{\nu\rho}\hat{T}^{c}{}_{\nu\lambda} - \frac{1}{e}e_{a}{}^{\rho}\mathcal{L}_{TEGR} + \frac{c^4}{8\pi G}\overset{\bullet}{\omega}{}^{c}{}_{a\sigma}\hat{S}_{c}{}^{\rho\sigma}. \tag{A.0.21}
$$

# Appendix B

# Universality of geodesics in metric theories

We will start exposing the discussion held by Ehlers, Pirani and Schild (EPS) in their article "*The geometry of free-fall and light propagation*" [58]. In this work, the authors claimed that if one wants to give a constructive set of axioms in order to describe the free-fall geometry, then the chronometric approach, which makes use of particles and clocks as basic concepts and the metric as the fundamental structure, does not seem particularly suitable. This arguments is based on the following points:

- it is difficult to deduce the Riemann form of the spacetime interval ($ds = \sqrt{g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}}$) solely from the behaviour of clocks without using light signals, which means one cannot easily justifiy why this form should be chosen over other possible forms, like the Newtonian one;
- if the metric components $g_{\mu\nu}$ are defined only through the chronometric hypotesis, they lack physical motivation and appear as unexplained assumptions rather than results derived from deeper principles, making them seem arbitrary;
- once the geodesic hypoteses, that freely falling particles and light rays follow goedesics, are accepted, one can already construct clocks using these notions, so the chronometric axiom becomes redundant.

For these reasons, EPS rejected clocks as basic tools for setting up the space geometry and proposed to use light rays and freely falling particles instead.
In their article, EPS mainly discussed how it is possible to derive the Lorentzian geometry, that underlies the spacetime of GR, from a compatible conformal and projective structures on a 4-dimensional manifold. They did this by enunciating some group of axioms, underlying spacetime geometry. The first group concerns its structure as a smooth manifold, and the

second the light propagation and conformal structure. The third group of axioms deals with the projective structure $\mathcal{P}$. We will summarize the content of these axioms in the following.

Before delving into physical aspects, we need some definitions.

**Definition B.0.1 (Conformal geometry).** A *conformal geometry* is given by an equivalence class $\mathcal{C}$ of metrics $g$ on a manifold $M$ and a connection $\Gamma$ compatible with $\mathcal{C}$. The equivalence relation that defines $\mathcal{C}$ is

$$g \overset{\mathcal{C}}{\equiv} g' \Longleftrightarrow \exists\ f \in M \ \text{ s.t. } \ g' = e^f g, \tag{B.0.1}$$

while the compatibility between $\mathcal{C}$ and $\Gamma$ reads as

$$\forall\ g \in \mathcal{C},\ \exists\ \text{1-form } \ \omega = \omega_\mu \mathrm{d}x^\mu \ \text{ s.t. } \ \nabla_\mu g_{\nu\rho} = \omega_\mu g_{\nu\rho}. \tag{B.0.2}$$

**Definition B.0.2 (Projective structure).** Two torsion-free affine connections on a manifold $M$ are called *projectively equivalent* if their geodesics coincides up to a reparametrization. Hence, a *projective structure* on a manifold $M$ is an equivalence class $\mathcal{P}$ of symmetric connections $\Gamma$, whose equivalence relation is

$$\Gamma \overset{\mathcal{P}}{\equiv} \Gamma' \Longleftrightarrow \exists\ \text{1-form } \ \omega \ \text{ s.t. } \ \Gamma'(A,B) = \Gamma(A,B) + A\omega(B) + B\omega(A). \tag{B.0.3}$$

A manifold with a projective structure is a *projective space* and a trasformation from a connection $\Gamma$ to another connection $\Gamma'$ is called *projective transformation.*

Roughly speaking, conformal geometry preserves angles, while projective geometry preserves straightness of paths (geodesics).

The motion of freely falling particles determine the geodesics of spacetime, hence they determine the projective structure $\mathcal{P}$.

Instead, the propagation of light determines at each point of the spacetime the infinitesimal null cones. A manifold in which the null cone has been singled out in the tangent space of each point, has a conformal structure $\mathcal{C}$ of Lorentzian signature[1]. $\mathcal{C}$ captures the causal structure of spacetime, but not distances or times (since those depend on the metric's overall scale, which $\mathcal{C}$ ignores by definition). This allows us to differentiate between time-like, space-like or null vectors. Moreover, the ratio of the lenghts of two non-null vectors at the same point is well-defined, as weel as the angles between two directions.

We can then define the *null projective geodesic* as a geodesic of $\mathcal{P}$ with null tangent vector.

[1]We can assume it is Lorentzian without loss of generality.

Having these ingredients and recalling that the conformal and projective structures of a pseudo-Riemannian spacetime determine the metric up to a constant [65], EPS showed that there is a sort of natural compatibility requirement, which states that

*"A conformal and a projective structures are said to be compatible if the null geodesics of the conformal structure*[2] *form a subset of the autoparallel paths of the projective structure."*

This condition is of a fundamental importance since it is necessary for the existence of a Lorentzian metric underlying both structure. EPS called a manifold with such a compatible pair $(\mathcal{C}, \mathcal{P})$ a *Weyl space*.
In a Weyl space, the conformal structure naturally provides a unique symmetric connection from the class of the affine connections which characterizes the compatible projective structure. This Weyl connection determines the parallel transport of vectors along a curve, which leaves unchanged the time-like, space-like or null character of a vector, and which leaves constant the ratio of the lenghts of two non-null vectors, as well as the angle between them. Geodesics, null or non-null, are completely characterized by the property that their tangential directions are parallely transported along them. Furthermore, an affine parameter can be determined along any geodesic. It is also possible to define the arc length $s$ along any non-null curve by parallely transporting any non-null vector $V$ along the curve ans defining the element of arc $\mathrm{d}s$ at different points as equal if they have the same ratio to the magnitude $|V|$ of $V$, i.e. $\mathrm{d}s/|V| = \text{constant}$.

**Definition B.0.3 (Weyl spacetime).** A *Weyl spacetime* is a physical spacetime which has a Weyl structure and where the proper time measured by a clock is the Weyl arc length along its world line.

In Weyl spacetimes, if two clocks, synchronized and identical at an event $A$, are separated and moved along different world lines $t$ anf $\bar{t}$ to the same event $B$, then not only will the elapsed time be different, $s \neq \bar{s}$ (first clock effect), but in general the two clocks will also tick at different rate at $B$, $\mathrm{d}s \neq \mathrm{d}\bar{s}$ (second clock effect, see Fig.B.1).

Finally, we arrive to the final piece of our journey: EPS provided a sketch of proof about the fact that a Weyl space reduces to a Riemann space. Technically, the affine connection of a Weyl space is a 1-form with values in the Lie algebra of the Lorentz group, namely $\omega \in \Omega^1(P, \mathfrak{so}(1,3))$. Its curvature is then a 2-form with values in the same Lie algebra. Consider now the holonomy group (see Def.(1.8.3)): as previously stated, it is always a subgroup of the structure group in which the Weyl connection takes values, namely $SO(1,3)^+$.

[2]Only null geodesics are invariant under conformal transformations.

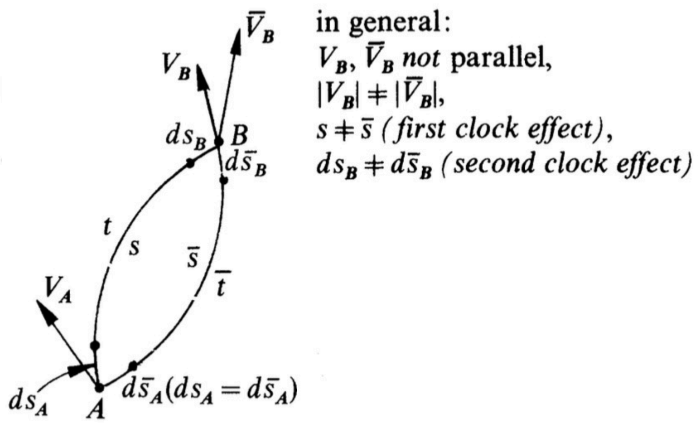
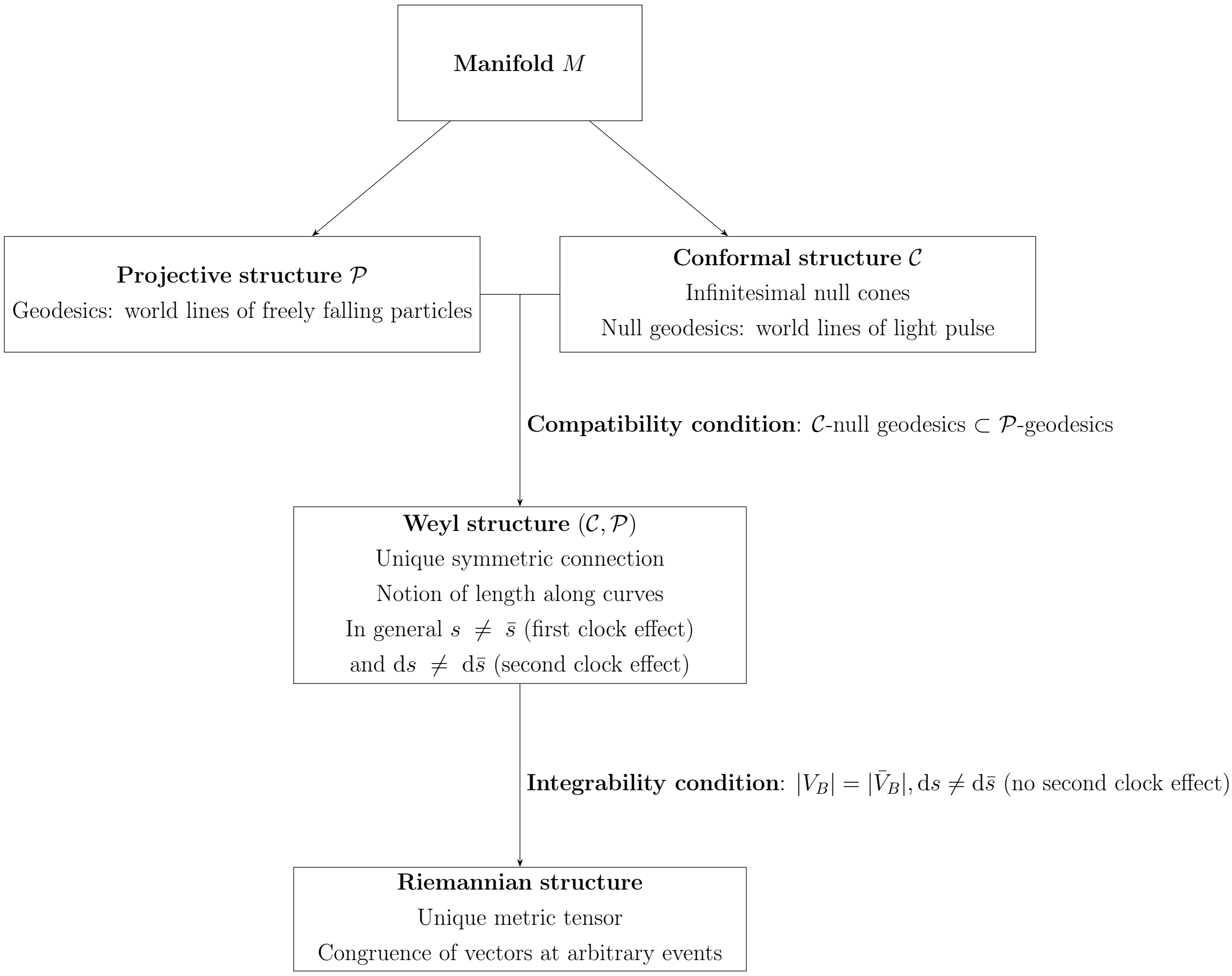



FIGURE B.1: In a Weyl space, in general, the parallel transport of a non-null vector from a point $A$ to a point $B$ along two paths $t$ and $\bar{t}$ is such that $V_B$ and $\bar{V}_B$ are not parallel and $|V_B| \neq |\bar{V}_B|$. Moreover, if we construct arc lenghts $s$ and $\bar{s}$ along $t$ and $\bar{t}$ as said previously, we have that $s \neq \bar{s}$ (first clock effect) and $\mathrm{d}s_B \neq \mathrm{d}\bar{s}_B$ (second clock effect).

From the holonomy theorems, we know that if the holonomy group is smaller than the full $GL(4, \mathbb{R})$, then the connection preserves some additional geometric structure. This allows one to construct a reduction of the frame bundle to a subgroup of the Lorentz group, which is known to be equivalent to the existence of a Lorentzian metric.
Hence, under parallel transport the magnitude of a vector is now always path independent, i.e. $|V_B| = |\bar{V}_B|$, meaning that there is no second clock effect anymore. This is called *integrability condition* and can also be expressed by requiring that the Ricci tensor is symmetric.

Thus, if GR provides the correct description of nature, then the projective structure of spacetime, explored by experiments with freely falling particles, and the conformal structure, explored by experiments with light propagation, automatically satisfty the compatibility requirement and the integrability condition. Hence, **the Riemannian spacetime structure can be fully explored by observing the world lines of particles in free fall and propagation of light**.

It is possible to summarize the steps done in this section with the following diagram:


**Manifold** $M$
**Projective structure** $\mathcal{P}$
Geodesics: world lines of freely falling particles
**Conformal structure** $\mathcal{C}$
Infinitesimal null cones
Null geodesics: world lines of light pulse
**Compatibility condition**: $\mathcal{C}$-null geodesics $\subset$ $\mathcal{P}$-geodesics
**Weyl structure** $(\mathcal{C}, \mathcal{P})$
Unique symmetric connection
Notion of length along curves
In general $s \neq \bar{s}$ (first clock effect)
and $\mathrm{d}s \neq \mathrm{d}\bar{s}$ (second clock effect)
**Integrability condition**: $|V_B| = |\bar{V}_B|, \mathrm{d}s \neq \mathrm{d}\bar{s}$ (no second clock effect)
**Riemannian structure**
Unique metric tensor
Congruence of vectors at arbitrary events

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